Holomorphic Elliptic Geometry
and Group Actions
CUSO minicourse and SMS Autumn Conference

University of Bern
September 5–7, 2018

1 Schedule

The minicourse will be given in the morning session, the scientific talks in the afternoon session. They take place in lecture room 106 in the main building (“Hauptgebäude”).

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<tr>
<td>09:00-10:00</td>
<td>mini-course</td>
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<td>10:00-10:30</td>
<td>coffee break</td>
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<td>10:30-11:30</td>
<td>mini-course</td>
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<td>11:30-13:30</td>
<td>lunch</td>
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<td>14:00-15:00</td>
<td>Kutzschebauch</td>
<td>Wold</td>
<td>Blanc</td>
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<td>15:00-16:00</td>
<td>Fornæss</td>
<td>Kraft</td>
<td>Bracci</td>
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<td>16:00-16:30</td>
<td>coffee break</td>
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<td>16:30-17:30</td>
<td>Simon</td>
<td>Dubouloz</td>
<td>Forstnerič</td>
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<td>conference dinner</td>
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2 Minicourse:
Introduction to compact and reductive group actions on affine algebraic varieties and Stein manifolds

by Gerald Schwarz, Brandeis University

Abstract: There has been much research of late on actions of reductive complex Lie groups on algebraic varieties. Analogous results have then been established for holomorphic actions of compact Lie groups and reductive complex Lie groups on Stein manifolds (or even Stein spaces). We recall the various results in the algebraic case and the analogous results in the holomorphic case. Some of the key terms are quotient spaces, Luna stratification, Luna’s slice theorem, moment mapping, strictly plurisubharmonic function and Kempf–Ness set. We will recall the definitions of all the relevant concepts and give many examples and applications.

3 Talks

Cremona groups

by Jérémy Blanc, University of Basel

Abstract: I will give a survey talk about the Cremona groups, which are groups of bimeromorphic transformations of the $n$-dimensional complex space (affine or projective). In dimension 1, there is not much to be said as we get the group of Möbius transformations. In dimension 2, the situation is much more rich, but we have a good knowledge and many tools that can be used. I will then describe the higher dimension, much more complicated.
Gromov topology and extension of biholomorphisms

by Filippo Bracci, Università di Roma Tor Vergata

Abstract: In dimension one, by the Carathéodory extension theorem, every Riemann map from the unit disc to a convex domain extends as a homeomorphism up to the boundary (when taking the Euclidean end compactification of the target domain, if unbounded). One abstract way to see this result, is to say that every Riemann from the unit disc to a simply connected domain extends as a homeomorphism in the Carathéodory topology up to the prime end boundary, and for a convex domain (different from $\mathbb{C}$) the Carathéodory topology is equivalent to the Euclidean topology of its closure. Carathéodory topology works well for quasi-conformal mappings in any dimension, but in higher dimension biholomorphisms are in general not quasi-conformal. Convex domains in the complex plane (different from $\mathbb{C}$) have the following features: they are (Kobayashi) hyperbolic and they are Gromov hyperbolic with respect to the hyperbolic metric.

In this talk I will outline the following result I recently proved with H. Gaussier and A. Zimmer: for a convex domain in $\mathbb{C}^n$ which is Kobayashi hyperbolic and Gromov hyperbolic with respect to the Kobayashi metric, the Gromov closure of the domain (endowed with the Gromov topology) is homeomorphic in a natural way to the Euclidean end compactification of the domain. There are many known examples of such convex domains.

It was known since Balogh and Bonk’s 2000 result that for a $C^2$-strongly pseudoconvex bounded domain the Gromov closure of the domain (endowed with the Gromov topology) is homeomorphic to the Euclidean closure. Moreover, quasi-isometries (in particular biholomorphisms) are homeomorphisms for the Gromov topology. Therefore, as a consequence, we have the following extension result: let $D$ be either a $C^2$-strongly pseudoconvex bounded domain, or a convex domain which is Kobayashi hyperbolic and Gromov hyperbolic with respect to the Kobayashi metric and let $D'$ be any convex domain. Then every biholomorphism (or, more generally, every quasi-isometry) $F: D \to D'$ extends as a homeomorphism to the Euclidean end compactification of $D$ and $D'$.

The proof of the result is, quite surprisingly, based on properties of commuting semigroups of holomorphic self-maps and allows to obtain new results about iteration and Denjoy–Wolff points in convex domains (possibly unbounded, and with no smoothness of the boundary).
Circle actions on real affine varieties

by Adrien Dubouloz, Université de Bourgogne

Abstract: By a classical pioneering result of Demazure, every normal complex affine surface endowed with an effective algebraic action of the complex multiplicative group is determined by a Weil divisor with rational coefficients on a suitable rational quotient of the surface. This kind of description was generalized by Altmann and Hausen to an algebro-combinatorial description of all normal affine varieties endowed with a faithful action of a split torus. Roughly, these can again be described by certain Weil divisors on suitable rational quotients obtained as a limit of GIT quotients, but whose coefficients are no longer rational numbers but instead rational polyhedra with vertices in the character lattice of the torus.

In order to tackle the case of non-split tori in general, one would have in principle to study in the framework of Galois descent the behavior of these data under the natural actions of the Galois group of a Galois extension of the base field which splits the given torus.

In this talk, I will explain and illustrate this strategy on a particularly simple case of independent geometric interest: algebraic actions of the real circle – that is, the “compact” real form of the multiplicative group – on geometrically normal real affine varieties. (Joint work in progress with Alvaro Liendo, University of Talca.)

Entire transcendental functions in one complex dimension

by John Erik Fornæss, Norwegian University of Science and Technology

Abstract: I will discuss some recent results on entire functions in one complex dimension.

Runge tubes

by Franc Forstnerič, University of Ljubljana

Abstract: We give a simple proof of the existence and plenitude of Runge tubes in $\mathbb{C}^n$ ($n > 1$) and, more generally, in any Stein manifold with the density property. We show in particular that for any affine algebraic submanifold $A$ of codimension at least two in a complex Euclidean space $\mathbb{C}^n$, the normal bundle of $A$ admits a holomorphic embedding onto a Runge domain in $\mathbb{C}^n$ which agrees with the inclusion map $A \hookrightarrow \mathbb{C}^n$ on the zero section. (Joint work with Erlend Fornæss Wold, University of Oslo.)
Small $G$-Varieties
by Hanspeter Kraft, University of Basel

Abstract: It is a hopeless task to classify complex SL$_2$-varieties (i.e. varieties with a non-trivial regular action of SL$_2$) if the dimension is greater than 3. Even dimension 3 is not an easy task, as these objects include all SL$_2$-embeddings, and the whole machinery of the Luna–Vust embedding theory comes into play. Thus the following result comes as a little surprise.

Theorem: Let $n \geq 5$. Then a smooth affine SL$_n$-variety of dimension $< 2n$ has the structure of an SL$_n$-vector bundle with fiber the standard representation or its dual.

Similar results hold for the simple groups Sp$_n$, and in some generalised way also for $E_6$, $E_7$ and $E_8$. We will explain the reason for this, and we will show why we cannot expect such results for the other simple groups, like $G_2$, SL$_3$, SL$_4$ and the orthogonal groups. (Joint work with Susanna Zimmermann and Andriy Regeta.)

Introductory Talk: What is Holomorphic Elliptic Geometry?
by Frank Kutzschebauch, University of Bern

An Example on $s$-H-Convexity in $\mathbb{C}^2$
by Lars Simon, Norwegian University of Science and Technology

Abstract: We construct a bounded domain $\Omega$ in $\mathbb{C}^2$ with boundary of class $C^{1,1}$, such that $\overline{\Omega}$ has a Stein neighborhood basis, but is not $s$-H-convex for any real number $s \geq 1$. (Joint work with Berit Stensønes, Norwegian University of Science and Technology.)

An Embedding of the unit ball in $\mathbb{C}^3$ whose image is not Runge in any strictly larger domain that contains it
by Erlend Fornæss Wold, University of Oslo

Abstract: We will explain the construction of an embedding as described in the title, and explain how it settles a central problem in geometric function theory in higher dimensions (in dimension three and higher).