Strength and polynomial functors

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The rank of infinite-by-infinite matrices

**Definition:** The rank of an \( \mathbb{N} \times \mathbb{N} \) matrix \( A \) is

\[
\text{rk}(A) := \sup \{ \text{rk}(B) \mid \text{finite submatrices } B \text{ of } A \} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}
\]

**Examples:**

1. The rank of the matrix

\[
\begin{pmatrix}
1 & * & \cdots & \cdots & \cdots \\
0 & 1 & * & \cdots & \cdots \\
\vdots & 0 & 1 & * & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
\]

is \( \infty \).

2. For linearly independent subsets \( \{v_1, \ldots, v_k\}, \{w_1, \ldots, w_k\} \subseteq \mathbb{C}^\mathbb{N} \) the matrix \( v_1 w_1^T + \cdots + v_k w_k^T \) has rank \( k \).
The rank of infinite-by-infinite matrices

Lemma: 
\[ A \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \text{ has rank } \leq k < \infty \iff A = \sum_{i=1}^{k} v_i w_i^T \text{ with } v_i, w_i \in \mathbb{C}^{\mathbb{N}} \]

Proof. Assume \( A \) has rank \( k \). Then \( A \) has a invertible \( k \times k \) submatrix. Permute the columns of \( A \) so that the first \( k \) columns of \( A \) are linearly independent. Call these first \( k \) columns \( v_1, \ldots, v_k \). To show that 
\[ A = \sum_{i=1}^{k} v_i w_i^T \]

for some \( w_1, \ldots, w_k \in \mathbb{C}^{\mathbb{N}} \), we need to show that every column of \( A \) is a linear combination of \( v_1, \ldots, v_k \). Let \( v \) be another column of \( A \). Then every finite submatrix of \( (v \ v_1 \ldots \ v_k) \) has rank \( \leq k \). Consider the vector space \( V_n = \{ \lambda \in \mathbb{C}^{k+1} \mid \text{pr}_n(\lambda_0 v - \lambda_1 v_1 + \cdots + \lambda_k v_k) = 0 \} \neq 0 \). We have \( V_{n+1} \subseteq V_n \) for all \( n \). It follows that \( V = \bigcap_n V_n \neq 0 \). Any nonzero element of \( V \) expresses \( v \) as a linear combination of \( v_1, \ldots, v_k \). \( \square \)
The rank of infinite-by-infinite matrices

**Fact:** An \( n \times m \) matrix \( A \) has rank \( \min(n,m) \) \( \iff \) \( \text{GL}_n \cdot A \cdot \text{GL}_m = \mathbb{C}^{n \times m} \)

**Theorem:** An \( \mathbb{N} \times \mathbb{N} \) matrix \( A \) has rank \( \infty \) \( \iff \) \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty = \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \)

**Proof.** (\( \iff \)) If the matrix \( A \) has rank \( k < \infty \), then \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty \) is contained in \{matrices in \( \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \) of rank \( \leq k \} \subsetneq \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \).

(\( \Rightarrow \)) Suppose \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty \subsetneq \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \). Then there is a nonzero equation on \( \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \) that is zero on \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty \). This equation uses only finitely many entries. So the rank of a particular finite submatrix has to be non-maximal for every element in \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty \). In particular, this is true for a permutations of \( A \). So the rank of \( A \) must be finite. \( \square \)

**Corollary:** Let \( A \) be an \( \mathbb{N} \times \mathbb{N} \) matrix. Then either \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty \) is dense in \( \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \) or \( A = \sum_{i=1}^{k} v_i w_i^T \) with \( v_1, \ldots, v_k, w_1, \ldots, w_k \in \mathbb{C}^\mathbb{N} \).
Similar theorems

**Definition:** The rank of a tuple of $\mathbb{N} \times \mathbb{N}$ matrices $(A_1, \ldots, A_k)$ is

$$\text{rk}(A_1, \ldots, A_k) := \inf \{ \text{rk}(\lambda_1 A_1 + \cdots + \lambda_k A_k) \mid (\lambda_1 : \cdots : \lambda_k) \in \mathbb{P}^{k-1} \}$$

**Theorem (Draisma-Eggermont)**

$$\text{rk}(A_1, \ldots, A_k) = \infty \iff \text{GL}_\infty \cdot (A_1, \ldots, A_k) \cdot \text{GL}_\infty = (\mathbb{C}^{\mathbb{N} \times \mathbb{N}})^k$$

**Definition:** The $q$-rank of a series

$$f = a_{111} x_1^3 + a_{112} x_1^2 x_2 + \cdots + a_{ijk} x_i x_j x_k + \ldots$$

is the minimal $k \leq \infty$ such that $f = \ell_1 q_1 + \cdots + \ell_k q_k$ with $\deg(\ell_i) = 1$.

**Theorem (Derksen-Eggermont-Snowden)**

$$\text{qrk}(f) = \infty \iff \text{GL}_\infty \cdot f = \{\text{all polynomial series of degree 3}\}$$
Similar theorems

Take $d \geq 2$.

**Definition** (Ananyan-Hochster)
The strength of a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]_{(d)}$ is the minimal $k$ such that

$$f = g_1 h_1 + \cdots + g_k h_k$$

with $g_1, \ldots, g_k, h_1, \ldots, h_k \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous of degree $< d$.

**Theorem** (B-Draisma-Eggermont)
For every $n$, let $X_n \subseteq \mathbb{C}[x_1, \ldots, x_n]_{(d)}$ be a closed subset such that:

(* ) We have $f \circ \ell \in X_m$ for all $f \in X_n$ and all linear maps $\ell: \mathbb{C}^m \to \mathbb{C}^n$.

Then either $X_n = \mathbb{C}[x_1, \ldots, x_n]_{(d)}$ for all $n \geq 0$ or there is a $k < \infty$ such that $\text{str}(f) \leq k$ for all $f \in X_n$ and $n \geq 0$. 
The semiring of functors \( P : Vec \rightarrow Vec \)

**Definition:** A functor \( P : Vec \rightarrow Vec \) sends

\[
V \mapsto P(V) \\
(\ell : V \rightarrow W) \mapsto (P(\ell) : P(V) \rightarrow P(W))
\]

such that \( P(id_V) = id_{P(V)} \) and \( P(\varphi \circ \psi) = P(\varphi) \circ P(\psi) \).

**Examples:** Take \( U \in Vec \) fixed.

- \( C_U : V \mapsto U, \ell \mapsto id_U \)
- \( T : V \mapsto V, \ell \mapsto \ell \)

You can add and multiply two functors \( P, Q : Vec \rightarrow Vec \).

\[
(P \oplus Q)(V) = P(V) \oplus Q(V), \quad (P \otimes Q)(V) = P(V) \otimes Q(V)
\]

**Definition:** The functor \( Q \) is a subfunctor of \( P \) when \( Q(V) \subseteq P(V) \) and \( Q(\ell) = P(\ell)|_{Q(\ell)} \). In his case, we have the functor \( V \mapsto P(V)/Q(V) \).
Polynomial functors as polynomials

**Definition:** The class of polynomial functors is the minimal class of functors $\text{Vec} \to \text{Vec}$ containing $T$ and all $C_U$ that is closed under addition, multiplication and taking subfunctors and quotients.

**Examples**

- Constants: $V \mapsto U$ for $U \in \text{Vec}$ fixed.
- Linear functors: $V \mapsto U \otimes V$ for $U \in \text{Vec}$ fixed.
- Matrices: $V \mapsto V \otimes V$
- Tensors: $V \mapsto V \otimes \cdots \otimes V$
- Polynomials: $V \mapsto S^d V$

**Remark:** The semiring of polynomial functors is graded.
Polynomial functors as topological spaces

**Definition:** Let $P, Q$ be polynomial functors. A morphism $\alpha : Q \to P$ is a family $(\alpha_V : Q(V) \to P(V))_{V \in \text{Vec}}$ of polynomial maps such that

$$
\begin{array}{ccc}
Q(V) & \xrightarrow{\alpha_V} & P(V) \\
\downarrow Q(\ell) & & \downarrow P(\ell) \\
Q(W) & \xrightarrow{\alpha_W} & P(W)
\end{array}
$$

commutes for all linear maps $\ell : V \to W$.

**Definition:** A closed subset $X \subseteq P$ sends

$$V \mapsto \text{closed subset } X(V) \subseteq P(V)$$

such that $P(\ell)(X(V)) \subseteq X(W)$ for all linear maps $\ell : V \to W$. 
The dichotomy

Let $P, Q$ be polynomial functors. Write $Q < P$ when $Q(d)$ is a quotient of $P(d)$ where $d$ is maximal with $Q(d) \not\cong P(d)$.

**Theorem** (B-Draisma-Eggermont-Snowden)
Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are polynomial functors $Q_1, \ldots, Q_k < P$ and $\alpha_i: Q_i \to P$ such that $X \subseteq \bigcup_i \text{im}(\alpha_i)$.

**Examples**

- \{matrices of rank $\leq k\} = \{v_1w_1^T + \cdots + v_kw_k^T \mid v_i, w_i \text{ vectors}\}$
- \{degree-$d$ polynomials that are zero on a codim-$k$ subspace\} = \{\ell_1g_1 + \cdots + \ell_kg_k \mid \deg(\ell_i) = 1, \deg(g_i) = d - 1\}
Applications

The dichotomy can be used to prove all the previous theorems.

**Theorem** (Draisma)
Every descending chain $P \supseteq X_1 \supseteq X_2 \supseteq \ldots$ of closed subsets stabilizes.

**Proof.** Using induction on $P$: take $Q_1, \ldots, Q_k < P$ and $\alpha_i : Q_i \to P$ such that $X_1 \subseteq \bigcup_i \text{im}(\alpha_i)$ and pull back the chain of closed subsets along each $\alpha_i$. The resulting chains all have to stabilize. □

**Theorem** (B-Draisma-Eggermont-Snowden)
Let $X \subseteq Q$ be a constructible subset and let $\alpha : Q \to P$ be a morphism. Then $\alpha(X)$ is constructible.

More analogues of results from finite-dimensional algebraic geometry?

Thank you for your attention!
References


