Strength and polynomial functors

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Infinite vectors and matrices

Let $K$ be an algebraically closed field of characteristic 0.

**Definition:**
(1) An infinite vector is a map $v: \mathbb{N} \rightarrow K$.
(2) An infinite matrix is a map $A: \mathbb{N} \times \mathbb{N} \rightarrow K$.

We write $v(i) = v_i$, $A(i, j) = A_{ij}$ and

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \ldots \\ A_{21} & A_{22} & A_{23} & \ldots \\ A_{31} & A_{32} & A_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
The rank of infinite matrices

**Definition:** The rank of an infinite matrix $A \in K^\mathbb{N} \times \mathbb{N}$ is

$$\text{rk}(A) := \sup \{ \text{rk}(B) \mid \text{finite submatrices } B \text{ of } A \} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

**Examples:**
(1) The ranks of the matrices

$$I_\infty = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

and

$$\begin{pmatrix} g \\ I_\infty \end{pmatrix}$$

for $g \in \text{GL}_n$ are $\infty$.

(2) For non-zero infinite vectors $v, w \in K^\mathbb{N}$, the infinite matrix $vw^T$ given by $(vw^T)_{ij} = v_i w_j$ has rank 1.
The rank of infinite matrices

Proposition:

$A \in K^{\mathbb{N} \times \mathbb{N}}$ has rank $\leq k < \infty \iff A = \sum_{j=1}^{k} v_j w_j^T$ with $v_j, w_j \in K^{\mathbb{N}}$

Proof. The direction $\iff$ is easy.
The rank of infinite matrices

Proposition: \( A \in K^{\mathbb{N} \times \mathbb{N}} \) has rank \( \leq k < \infty \) \( \iff \) \( A = \sum_{j=1}^{k} v_j w_j^T \) with \( v_j, w_j \in K^\mathbb{N} \)

Proof. The direction \( \iff \) is easy.

For \( \Rightarrow \), assume for convenience that both \( A \) and its topleft \( k \times k \) submatrix have rank \( k \). Let \( v_1, \ldots, v_k \in K^{\mathbb{N}} \) be the first \( k \) columns of \( A \).

Goal: prove that every column of \( A \) is a linear combination of \( v_1, \ldots, v_n \).
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Let $v'$ be another column of $A$ and take

$$V_n := \{ (\lambda', \lambda_1, \ldots, \lambda_k) \in K^{k+1} \mid \forall i \leq n : \lambda' v_i' = \lambda_1 v_{1i} + \cdots + \lambda_k v_{ki} \}$$

We have $V_{n+1} \subseteq V_n$ and $V_n \neq 0$ for all $n \in \mathbb{N}$. 
The rank of infinite matrices

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\( A \in K^{\mathbb{N} \times \mathbb{N}} \) has rank \( \leq k < \infty \) \( \iff \) \( A = \sum_{j=1}^{k} v_j w_j^T \) with \( v_j, w_j \in K^{\mathbb{N}} \)

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\]
We have \( V_{n+1} \subseteq V_n \) and \( V_n \neq 0 \) for all \( n \in \mathbb{N} \).
\( \Rightarrow \) \( \bigcap_{n \in \mathbb{N}} V_n \neq 0 \)

Take \( (1, \lambda_1, \ldots, \lambda_k) \in \bigcap_{n \in \mathbb{N}} V_n \). Then \( v' = \lambda_1 v_1 + \cdots + \lambda_k v_k \). \( \square \)
The Zariski topology on $K^\mathbb{N} \times \mathbb{N}$

**Definition:** A polynomial function on $K^\mathbb{N} \times \mathbb{N}$ sends a matrix $A$ to a finite polynomial expression of its entries $A_{ij}$.

**Example:** $f(A) = A_{11}^3 A_{22} - A_{12} A_{21}$

**Nonexample:** $f(A) = A_{11}^2 + A_{22}^2 + A_{33}^3 + \ldots$
The Zariski topology on $K^N \times N$

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**Nonexample:** $f(A) = A_{11}^2 + A_{22}^2 + A_{33}^3 + \ldots$

**Definition:** A subset of $K^N \times N$ is Zariski-closed when it is of the form

\[
\{ A \in K^N \times N \mid f(A) = 0 \text{ for all } f \in S \}
\]

where $S$ is a set of polynomial functions on $K^N \times N$.

**Example:** Take $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then the set

\[
\{ A \in K^N \times N \mid \text{rk}(A) \leq k \}
\]

is Zariski-closed.
The rank of infinite-by-infinite matrices

**Fact:** An $n \times m$ matrix $A$ has rank $\min(n, m) \iff \overline{\text{GL}_n \cdot A \cdot \text{GL}_m} = K^{n \times m}$

**Theorem:** A matrix $A \in K^{\mathbb{N} \times \mathbb{N}}$ has rank $\infty \iff \overline{\text{GL}_\infty \cdot A \cdot \text{GL}_\infty} = K^{\mathbb{N} \times \mathbb{N}}$
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**Proof.** ($\Leftarrow$) If $\text{rk}(A) = k < \infty$, then
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\overline{\text{GL}_\infty \cdot A \cdot \text{GL}_\infty} \subseteq \{ A \in K^{\mathbb{N} \times \mathbb{N}} \mid \text{rk}(A) \leq k \} \subsetneq K^{\mathbb{N} \times \mathbb{N}}
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$$GL_\infty \cdot A \cdot GL_\infty \subseteq \{ A \in K^{\mathbb{N} \times \mathbb{N}} \mid \text{rk}(A) \leq k \} \not\subseteq K^{\mathbb{N} \times \mathbb{N}}$$

($\Rightarrow$) Suppose $GL_\infty \cdot A \cdot GL_\infty \not\subseteq K^{\mathbb{N} \times \mathbb{N}}$. Then $f(GL_\infty \cdot A \cdot GL_\infty) = 0$ for some nonzero polynomial function $f$.

The function $f$ uses only finitely many entries.
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The function $f$ uses only finitely many entries.

$\Rightarrow$ The rank of a particular finite submatrix has to be non-maximal for every element in $\text{GL}_\infty \cdot A \cdot \text{GL}_\infty$.

$\Rightarrow$ The rank of a particular finite submatrix has to be non-maximal for every permutation of $A$.

$\Rightarrow$ The rank of $A$ must be finite.
The rank of infinite-by-infinite matrices

Let \( A \in K^{\mathbb{N} \times \mathbb{N}} \) be an infinite matrix.

**Proposition:**
The matrix \( A \) has rank \( \leq k < \infty \) \( \iff \) \( A = \sum_{j=1}^{k} v_j w_j^T \) with \( v_j, w_j \in K^{\mathbb{N}} \)

**Theorem:**
The matrix \( A \) has rank \( \infty \) \( \iff \) \( \overline{\text{GL}_\infty \cdot A \cdot \text{GL}_\infty} = K^{\mathbb{N} \times \mathbb{N}} \)
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Let $A \in K^\mathbb{N} \times \mathbb{N}$ be an infinite matrix.

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The matrix $A$ has rank $\infty \iff \overline{\text{GL}_\infty \cdot A \cdot \text{GL}_\infty} = K^\mathbb{N} \times \mathbb{N}$

**Corollary:** Precisely one of the following holds:
(1) $\text{GL}_\infty \cdot A \cdot \text{GL}_\infty$ is dense in $K^\mathbb{N} \times \mathbb{N}$.
(2) $A = \sum_{j=1}^{k} v_j w_j^T$ with $v_j, w_j \in K^\mathbb{N}$.

**Remark:** Similar statements hold for:
(1) Tuples of matrices (Draisma, Eggermont)
(2) Homogeneous polynomials (B, Draisma, Eggermont)
(3) Tensors (B, Draisma, Eggermont)
**Similar statements**

**Definition:** The rank of a tuple of matrices \((A_1, \ldots, A_k)\) is

\[
\text{rk}(A_1, \ldots, A_k) := \inf\{\text{rk}(\lambda_1 A_1 + \cdots + \lambda_k A_k) \mid (\lambda_1 : \cdots : \lambda_k) \in \mathbb{P}^{k-1}\}
\]

**Definition:** The strength of a homogeneous polynomial \(f\) of degree \(d \geq 2\) is the minimal \(k \leq \infty\) such that \(f = g_1 h_1 + \cdots + g_k h_k\) with \(\deg(g_i), \deg(h_i) < d\).

**Definition:** The flattening rank of a \(d\)-way tensor \(t\) is the minimal \(k \leq \infty\) such that \(t = f_1 + \cdots + f_k\) with each tensor \(f_i\) has some rank-1 flattening.
Why look at infinite objects?

Let $A \in K^\mathbb{N} \times \mathbb{N}$ be an infinite matrix.

**Corollary**: Precisely one of the following holds:

1. $\text{GL}_\infty \cdot A \cdot \text{GL}_\infty$ is dense in $K^\mathbb{N} \times \mathbb{N}$.
2. $A = \sum_{j=1}^{k} v_j w_j^T$ with $v_j, w_j \in K^\mathbb{N}$.

Let $X \subsetneq K^\mathbb{N} \times \mathbb{N}$ be a $(\text{GL}_\infty \times \text{GL}_\infty)$-stable Zariski-closed subset.

$\Rightarrow \text{rk}(X) \leq k$ for some $k < \infty$. 
Why look at infinite objects?

Let \( A \in K^{\mathbb{N} \times \mathbb{N}} \) be an infinite matrix.

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Let \( X \subseteq K^{\mathbb{N} \times \mathbb{N}} \) be a \((\text{GL}_\infty \times \text{GL}_\infty)\)-stable Zariski-closed subset.
\[ \Rightarrow \text{rk}(X) \leq k \text{ for some } k < \infty. \]

Let \( X_n \subseteq K^{\mathbb{N} \times \mathbb{N}} \) be the projection on \( K^{n \times n} \).
\[ \Rightarrow \text{rk}(B) \leq k \text{ for all } B \in X_n. \]
\[ \Rightarrow \text{Matrices } B \in X_n \text{ can always be expressed using } 2k \text{ vectors.} \]

**Remark:** The bound \( k \) does not depend on \( n \).
Categories and functors

**Definition:** A category $C$ has objects $C, D \in C$, morphisms $C \to D$ and compositions. Taking compositions is associative and for every object $C \in C$ there is an identity $\text{id}_C : C \to C$.

**Examples:**
(0) The category $\text{Set}$. Objects are sets and morphisms are maps.
(1) The category $\text{Vec}$. Objects are finite-dimensional vector spaces and morphisms are linear maps.
(2) The category $\text{Top}$. Objects are topological spaces and morphisms are continuous maps.
(3) For $k \in \mathbb{N}$, the category $\text{Vec}^k$. Objects are $k$-tuples $V = (V_1, \ldots, V_k)$ and morphisms are $k$-tuples $\ell = (\ell_1, \ldots, \ell_k)$. 
Categories and functors

Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition: A functor $F: \mathcal{C} \to \mathcal{D}$ assigns

- to every object $C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$
- to every morphism $\ell: C \to C'$ a morphism $F(\ell): F(C) \to F(C')$

such that $F(\ell \circ \ell') = F(\ell) \circ F(\ell')$ and $F(id_C) = id_{F(C)}$.

Examples:

(0) The functor $\text{For}: \text{Vec} \to \text{Set}$ with $\text{For}(V) = V$ and $\text{For}(\ell) = \ell$.
(1) The functor $\text{Zar}: \text{Vec} \to \text{Top}$ with $\text{Zar}(V) = V$ and $\text{Zar}(\ell) = \ell$.
(2) For $k \in \mathbb{N}$, the functor $\Delta: \text{Vec} \to \text{Vec}^k$ with $\Delta(V) = (V, \ldots, V)$ and $\Delta(\ell) = (\ell, \ldots, \ell)$. 
**Polynomial functors as polynomials**

\( \text{Vec}^k = \) category of \( k \)-tuples of finite-dimensional vector spaces.

**Definition:** A polynomial functor \( P : \text{Vec}^k \rightarrow \text{Vec} \)

1. assigns a vector space \( P(V) \in \text{Vec} \) to every \( V \in \text{Vec}^k \)
2. assigns a polynomial map

\[
\begin{align*}
\text{Mor}(V, W) & \rightarrow \text{Hom}(P(V), P(W)) \\
\ell & \mapsto P(\ell)
\end{align*}
\]

...to every pair \( (V, W) \in \text{Vec}^k \times \text{Vec}^k \)

such that \( P(\text{id}_V) = \text{id}_{P(V)} \) and \( P(\ell_1 \circ \ell_2) = P(\ell_1) \circ P(\ell_2) \).

**Examples:** Take \( U \in \text{Vec} \) fixed and \( i \in \{1, \ldots, k\} \).

1. Take \( C_U(V) = U \) for all \( V \in \text{Vec}^k \) and \( C_U(\ell) = \text{id}_U \) for all \( \ell \).
2. Take \( T_i(V) = V_i \) for all \( V \in \text{Vec}^k \) and \( C_U(\ell) = \ell_i \) for all \( \ell \).
Polynomial functors as polynomials

Let $P, Q$ be polynomial functors.

**Definition:** Define the direct sum $P \oplus Q$ by:

$$(P \oplus Q)(V) = P(V) \oplus Q(V) \text{ and } (P \oplus Q)(\ell)(v, w) = (P(\ell)(v), Q(\ell)(w))$$

**Definition:** Define the tensor product $P \otimes Q$ by:

$$(P \otimes Q)(V) = P(V) \otimes Q(V) \text{ and } (P \otimes Q)(\ell)(v \otimes w) = P(\ell)(v) \otimes Q(\ell)(w)$$

**Examples:**

1. $T \oplus T$ is the polynomial functor of 2-tuples of vectors.
2. $T_1 \otimes T_2$ is the polynomial functor of matrices.
3. $T_1 \otimes \cdots \otimes T_k$ is the polynomial functor of $k$-way tensors.
Polynomial functors as polynomials

Let $P, Q$ be polynomial functors.

**Definition:** The functor $Q$ is a subfunctor of $P$ when $Q(V) \subseteq P(V)$.

Suppose that $Q$ is a subfunctor of $P$.

**Definition:** Define the quotient $P/Q$ by $(P/Q)(V) = P(V)/Q(V)$.

**Examples:**

1. $T \otimes T$ has $S^2$ and $\bigwedge^2$ as subfunctors.
2. $T^\otimes_k := T \otimes \cdots \otimes T$ has $S^d$ as subfunctor.
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**Fact:** Every polynomial functor can be obtained from the constants $C_U$ and variables $T_1, \ldots, T_k$ using direct sums, tensor products, subfunctors and quotients.
Polynomial functors as topological spaces

**Definition:** A closed subset \( X \subseteq P \) assigns a closed subset 
\[
X(V) \subseteq P(V)
\]
to every \( V \in \text{Vec}^k \) such that \( P(\ell)(X(V)) \subseteq X(W) \) for all \( \ell: V \to W \).

**Examples:**
(1) A closed subset of \( C_U \) is a closed subset of \( U \).
(2) \( \{ \text{linearly dependent tuples of vectors} \} \subseteq T \oplus \cdots \oplus T \).
(3) \( \{ \text{matrices of rank} \leq r \} \subseteq T_1 \otimes T_2 \).
(4) \( \{ \text{tensors of rank} \leq r \} \subseteq T_1 \otimes \cdots \otimes T_k \).
(5) \( \{ \text{polynomials that are zero on a codim} \leq r \text{ subspace} \} \subseteq S^d \).
Polynomial functors as topological spaces

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**Examples:**
1. A closed subset of $C_U$ is a closed subset of $U$.
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4. $\{\text{tensors of rank} \leq r\} \subseteq T_1 \otimes \cdots \otimes T_k$.
5. $\{\text{polynomials that are zero on a codim} \leq r \text{ subspace}\} \subseteq S^d$.

**Remark:** For every $V \in \text{Vec}^k$, we have the action

$$\text{GL}(V) := \text{GL}(V_1) \times \cdots \times \text{GL}(V_k) \quad \to \quad \text{GL}(P(V))$$

$$\ell = (\ell_1, \ldots, \ell_k) \quad \mapsto \quad P(\ell)$$
Morphisms between polynomial functors

Let $P, Q$ be polynomial functors.

**Definition:** A polynomial transformation $\alpha : Q \to P$ is a family 

$$(\alpha_V : Q(V) \to P(V))_{V \in \text{Vec}_k}$$

of polynomial maps such that

$$Q(V) \xrightarrow{\alpha_V} P(V) \quad Q(\ell) \downarrow \quad \downarrow P(\ell)$$

$$Q(W) \xrightarrow{\alpha_W} P(W)$$

commutes for all $\ell : V \to W$.

**Example:** Take $P = T_1 \otimes T_2$ and $Q = T_1 \oplus T_1 \oplus T_2 \oplus T_2$. Then 

$$\alpha_{(V,W)} : V \oplus V \oplus W \oplus W \to V \otimes W$$

$$(v_1, v_2, w_1, w_2) \mapsto v_1 \otimes w_1 + v_2 \otimes w_2$$

defines an polynomial transformation $\alpha : Q \to P$. 
Main theorem

Let $P, Q$ be polynomial functors. Write $Q < P$ when $Q(d)$ is a quotient of $P(d)$ where $d$ is maximal with $Q(d) \not\cong P(d)$.

Dichotomy Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are polynomial functors $Q_1, \ldots, Q_k < P$ and $\alpha_i : Q_i \to P$ such that $X \subseteq \bigcup_i \text{im}(\alpha_i)$. 
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Examples

- \{matrices of rank \leq r\} = \{v_1 w_1^T + \cdots + v_r w_r^T \mid v_i, w_i \text{ vectors}\}
- \{degree-$d$ polynomials that are zero on a codim \leq r subspace\} = \{\ell_1 g_1 + \cdots + \ell_r g_r \mid \deg(\ell_i) = 1, \deg(g_i) = d - 1\}
Applications

**Theorem (Draisma)**
Every descending chain $P \supseteq X_1 \supseteq X_2 \supseteq \ldots$ of closed subsets stabilizes.
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Every descending chain $P \supsetneq X_1 \supsetneq X_2 \supsetneq \ldots$ of closed subsets stabilizes.

**Proof.** Using induction on $P$: take $Q_1, \ldots, Q_k < P$ and $\alpha_i : Q_i \to P$ such that $X_1 \subseteq \bigcup_i \text{im}(\alpha_i)$ and pull back the chain of closed subsets along each $\alpha_i$. The resulting chains all have to stabilize. \qed
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Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq Q$ be a constructible subset and let $\alpha: Q \to P$ be a morphism. Then $\alpha(X)$ is constructible.

More analogues from finite-dimensional affine algebraic geometry?
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Every descending chain $P \supseteq X_1 \supseteq X_2 \supseteq \ldots$ of closed subsets stabilizes.

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Thank you for your attention!

