The monic rank and instances of Shapiro’s Conjecture

Arthur Bik
University of Bern

j.w.w. Jan Draisma, Alessandro Oneto and Emanuele Ventura

MEGA 2019, Madrid, 20 June 2019
A Conjecture by Shapiro

Conjecture (Boris Shapiro)

Every homogeneous polynomial $f \in \mathbb{C}[x, y]$ of degree $d \cdot e$ is the sum of at most $d$ $d$-th powers of polynomials of degree $e$. 
A Conjecture by Shapiro

Conjecture (Boris Shapiro)

*Every homogeneous polynomial* \( f \in \mathbb{C}[x, y] \) *of degree* \( d \cdot e \) *is the sum of at most* \( d \) *\( d \)-th powers of polynomials of degree* \( e \).*

Why believe this?

- True when \( e = 1 \), when \( d = 1 \) and when \( d = 2 \).
- The projective variety

\[
\left\{ [h^d] \mid h \in \mathbb{C}[x, y]_e \right\} \subseteq \mathbb{P}(\mathbb{C}[x, y]_{d \cdot e})
\]

has dimension \( e \) in a projective space of dimension \( d \cdot e \).

\[\Rightarrow\] Its \( d \)-th secant variety is expected to be everything.

- True for \((d, e) = (3, 2)\) by Lundqvist, Oneto, Reznick and Shapiro.
Example: \( \{ \text{deg } d \} = \{ \text{sum of } d\text{-th powers of deg 1} \} \)

Consider

\[
(x + a_1y)^d + (x + a_2y)^d + \cdots + (x + a_dy)^d
\]

\[
= dx^d + \binom{d}{1} b_1 x^{d-1} y + \binom{d}{2} b_2 x^{d-2} y^2 + \cdots + \binom{d}{d} b_d y^d
\]

with \( b_k = a_1^k + \cdots + a_d^k \).
Example: \( \{ \text{deg } d \} = \{ \text{sum of } d\text{-th powers of deg } 1 \} \)

Consider
\[
(x + a_1y)^d + (x + a_2y)^d + \cdots + (x + a_dy)^d =
\]
\[
dx^d + \binom{d}{1} b_1 x^{d-1}y + \binom{d}{2} b_2 x^{d-2}y^2 + \cdots + \binom{d}{d} b_dy^d
\]
with \( b_k = a_1^k + \cdots + a_d^k \).

Fact (Hilbert): The map \((a_1, \ldots, a_d) \mapsto (b_1, \ldots, b_d)\) is a finite morphism.
Example: \( \{ \text{deg } d \} = \{ \text{sum of } d\text{-th powers of deg 1} \} \)

Consider

\[
(x + a_1y)^d + (x + a_2y)^d + \cdots + (x + a_dy)^d
\]

\[
= dx^d + \binom{d}{1} b_1x^{d-1}y + \binom{d}{2} b_2x^{d-2}y^2 + \cdots + \binom{d}{d} b_dy^d
\]

with \( b_k = a_1^k + \cdots + a_d^k \).

Fact (Hilbert): The map \((a_1, \ldots, a_d) \mapsto (b_1, \ldots, b_d)\) is a finite morphism.

Using coordinate transformations, this implies:

\[
\mathbb{C}[x, y]_{(d)} = \left\{ \ell_1^d + \cdots + \ell_d^d \mid \ell_1, \ldots, \ell_d \in \mathbb{C}[x, y]_{(1)} \right\}
\]
The monic rank

- $V$ a finite-dimensional vector space
- $X \subseteq V$ a non-degenerate irreducible Zariski-closed cone
- $h: V \to \mathbb{C}$ a non-zero linear function and $H = h^{-1}(1) \subseteq V$

**Definition**
The monic rank of a vector $v \in V \setminus h^{-1}(0)$ is the minimal $r$ such that

$$\frac{r}{h(v)} \cdot v = w_1 + \cdots + w_r$$

with $w_1, \ldots, w_r \in X \cap H$.

**Theorem**
monic rank $\leq 2 \cdot (\text{the generic monic rank}) < \infty$
Shapiro’s Conjecture (Monic Version)

Every $f \in \mathbb{C}[x, y]_{(d \cdot e)}$ with leading coefficient $d$ has monic rank $\leq d$.

$$X = \{d\text{-th powers of homogeneous polynomials of degree } e\}$$
Shapiro’s Conjecture (Monic Version)

Every $f \in \mathbb{C}[x, y]_{(d \cdot e)}$ with leading coefficient $d$ has monic rank $\leq d$.

$$X = \{d\text{-th powers of homogeneous polynomials of degree } e\}$$

**Goal:** We want to show that

$$\prod_{i=1}^{d} \{ f \in \mathbb{C}[x, y]_{(e)} \text{ monic} \} \rightarrow \mathbb{C}[x, y]_{(d \cdot e)}$$

$$(f_1, \ldots, f_d) \mapsto f_1^d + \cdots + f_d^d$$

is a finite morphism.
Shapiro’s Conjecture (Monic Version)

Every \( f \in \mathbb{C}[x, y]_{(d,e)} \) with leading coefficient \( d \) has monic rank \( \leq d \).

\[ X = \{d\text{-th powers of homogeneous polynomials of degree } e\} \]

**Goal:** We want to show that

\[
\prod_{i=1}^{d} \{ f \in \mathbb{C}[x, y]_{(e)} \text{ monic} \} \rightarrow \mathbb{C}[x, y]_{(d,e)}
\]

\[
(f_1, \ldots, f_d) \mapsto f_1^d + \cdots + f_d^d
\]

is a finite morphism.

**Proposition:** This is true if \((c_{ij})_{ij} = 0\) is the only solution of the equation

\[
dx^{de} = \sum_{i=1}^{d} \left( x^e + c_{i1}x^{e-1}y + \cdots + c_{ie}y^e \right)^d
\]
Assume that \((c_{ij})_{ij}\) satisfies

\[
dx^{de} = \sum_{i=1}^{d} \left( x^e + c_{i1} x^{e-1} y + \cdots + c_{ie} y^e \right)^d
\]
Reduction to a Gröbner basis computation

Assume that \((c_{ij})_{ij}\) satisfies

\[
dx^{de} = \sum_{i=1}^{d} \left( x^e + c_{i1} x^{e-1} y + \cdots + c_{ie} y^e \right)^d
\]

Case 1
We have \(c_{ie} = 0\) for all \(i\). Divide by \(x^d\).

\(\mapsto\) This replaces \(e\) by \(e - 1\).
Reduction to a Gröbner basis computation

Assume that \((c_{ij})_{ij}\) satisfies

\[
dx^{de} = \sum_{i=1}^{d} \left( x^e + c_{i1}x^{e-1}y + \cdots + c_{ie}y^e \right)^d
\]

Case 1
We have \(c_{ie} = 0\) for all \(i\). Divide by \(x^d\).

\[\implies\] This replaces \(e\) by \(e - 1\).

Case 2
After permuting summands and scaling \(y\), we get \(c_{1e} = 1\).

\[\implies\] A Gröbner basis can contradict this.
Reduction to a Gröbner basis computation

Assume that \((c_{ij})_{ij}\) satisfies

\[ dx^{de} = \sum_{i=1}^{d} \left( x^e + c_{i1}x^{e-1}y + \cdots + c_{ie}y^e \right)^d \]

**Case 1**
We have \(c_{ie} = 0\) for all \(i\). Divide by \(x^d\).
\[\rightarrow\] This replaces \(e\) by \(e - 1\).

**Case 2**
After permuting summands and scaling \(y\), we get \(c_{1e} = 1\).
\[\rightarrow\] A Gröbner basis can contradict this.

The computation finished for \((d, e) = (3, 2), (3, 3), (3, 4), (4, 2)\).
Conjecture (Boris Shapiro)

Every homogeneous polynomial \( f \in \mathbb{C}[x, y] \) of degree \( d \cdot e \) is the sum of at most \( d \) \( d \)-th powers of polynomials of degree \( e \).

Why believe this?

- True when \( e = 1 \), when \( d = 1 \) and when \( d = 2 \).
- The projective variety

\[
\{ [h^d] \mid h \in \mathbb{C}[x, y]_{(e)} \} \subseteq \mathbb{P}(\mathbb{C}[x, y]_{(d \cdot e)})
\]

has dimension \( e \) in a projective space of dimension \( d \cdot e \).

\( \Rightarrow \) Its \( d \)-th secant variety is expected to be everything.

- True for \((d, e) = (3, 2)\) by Lundqvist, Oneto, Reznick and Shapiro.
- True for \((d, e) = (3, 3), (3, 4), (4, 2)\).
Other examples of (monic) rank

Some other objects that have a rank:
- Matrices
- Symmetric matrices
- Trace-zero matrices
- Tensors

Question: What should be their "leading coefficient"s?
Natural choice: Let \( h \in \mathcal{V} \) be a highest weight vector.

Question: How do the maximal rank and monic rank compare?
Other examples of (monic) rank

Some other objects that have a rank:

- Matrices
- Symmetric matrices
- Trace-zero matrices
- Tensors

**Question**: What should be their "leading coefficient"s?
Other examples of (monic) rank

Some other objects that have a rank:

- Matrices
- Symmetric matrices
- Trace-zero matrices
- Tensors

**Question**: What should be their "leading coefficient"s?

**Natural choice**: Let $h \in V^*$ be a highest weight vector.
Other examples of (monic) rank

Some other objects that have a rank:

- Matrices (top-left entry)
- Symmetric matrices (top-left entry)
- Trace-zero matrices (top-right entry)
- Tensors (coefficient of $e_1 \otimes \cdots \otimes e_1$)

**Question**: What should be their "leading coefficient"s?

**Natural choice**: Let $h \in V^*$ be a highest weight vector.
Other examples of (monic) rank

Some other objects that have a rank:

- Matrices (top-left entry)
- Symmetric matrices (top-left entry)
- Trace-zero matrices (top-right entry)
- Tensors (coefficient of $e_1 \otimes \cdots \otimes e_1$)

**Question**: What should be their "leading coefficient"s?

**Natural choice**: Let $h \in V^*$ be a highest weight vector.

**Question**: How do the maximal rank and monic rank compare?
The space of $2 \times 2 \times 2$ tensors:

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left| \begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right. \right| a_{ij}, b_{ij} \in \mathbb{C} \right\}$$
**2 x 2 x 2 Tensors**

The space of $2 \times 2 \times 2$ tensors:

$$C^2 \otimes C^2 \otimes C^2 = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \mid a_{ij}, b_{ij} \in \mathbb{C} \right\}$$

The tensors of rank $\leq 1$:

$$X = \left\{ (A \mid B) \mid \text{rk}(A), \text{rk}(B) \leq 1, A, B \text{ are linearly dependent} \right\}$$

**Fact:** The maximal rank of a $2 \times 2 \times 2$ tensor is 3.
2 x 2 x 2 Tensors

The space of $2 \times 2 \times 2$ tensors:

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \middle| a_{ij}, b_{ij} \in \mathbb{C} \right\}$$

The tensors of rank $\leq 1$:

$$X = \left\{ (A \mid B) \middle| \begin{array}{c} \operatorname{rk}(A), \operatorname{rk}(B) \leq 1 \\ A, B \text{ are linearly dependent} \end{array} \right\}$$

Fact: The maximal rank of a $2 \times 2 \times 2$ tensor is 3.

Let $a_{11}$ be the leading coefficient. The maximal monic rank is $\geq 3$.

Question: Is the maximal monic rank equal to 3?
Orbits of tensors

We have 3 commuting actions of $\mathbb{C}$:

- $(v_1 \ v_2 \ | \ w_1 \ w_2) \mapsto (v_1 \ v_2 + \lambda v_1 \ | \ w_1 \ w_2 + \lambda w_1)$
- \[
\begin{pmatrix}
  r_1 \\
  r_2
\end{pmatrix} \begin{pmatrix}
  s_1 \\
  s_2
\end{pmatrix} \mapsto \begin{pmatrix}
  r_1 \\
  r_2 + \lambda r_1
\end{pmatrix} \begin{pmatrix}
  s_1 \\
  s_2 + \lambda s_1
\end{pmatrix}
\]
- $(A \ | \ B) \mapsto (A \ | \ B + \lambda A)$
Orbits of tensors

We have 3 commuting actions of $\mathbb{C}$:

- $(v_1 \ v_2 \ | \ w_1 \ w_2) \mapsto (v_1 \ v_2 + \lambda v_1 \ | \ w_1 \ w_2 + \lambda w_1)$

- \[
\begin{pmatrix}
  r_1 \\
  r_2 \\
\end{pmatrix}
\begin{pmatrix}
  s_1 \\
  s_2 \\
\end{pmatrix}
\mapsto
\begin{pmatrix}
  r_1 \\
  r_2 + \lambda r_1 \\
\end{pmatrix}
\begin{pmatrix}
  s_1 \\
  s_2 + \lambda s_1 \\
\end{pmatrix}
\]

- $(A \ | \ B) \mapsto (A \ | \ B + \lambda A)$

Remark: These operations do not change ranks or leading coefficients.
Orbits of tensors

We have 3 commuting actions of $\mathbb{C}$:

- $(v_1 v_2 \mid w_1 w_2) \mapsto (v_1 v_2 + \lambda v_1 \mid w_1 w_2 + \lambda w_1)$

- \[
\begin{pmatrix}
  r_1 \\ r_2
\end{pmatrix}
\begin{pmatrix}
  s_1 \\ s_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
  r_1 \\ r_2 + \lambda r_1
\end{pmatrix}
\begin{pmatrix}
  s_1 \\ s_2 + \lambda s_1
\end{pmatrix}
\]

- $(A \mid B) \mapsto (A \mid B + \lambda A)$

**Remark:** These operations do not change ranks or leading coefficients.

**Lemma:** Every $2 \times 2 \times 2$ tensor with a non-zero leading coefficient lies in the orbit of a tensor of the form

\[
\begin{pmatrix}
  c & 0 & 0 \\ 0 & \mu_3 & \mu_2 \\ 0 & \mu_1 & \lambda
\end{pmatrix}
\]

with $c, \lambda, \mu_1, \mu_2, \mu_3 \in \mathbb{C}$. 
Tensors with monic rank $\leq 2$

Claim: The set of sums of two monic tensors with rank 1 is

$$\mathbb{C}^3 \cdot \left\{ \begin{pmatrix} 2 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 \\ \mu_2 & 0 \end{pmatrix} \mid \mu_1, \mu_2, \mu_3 \in \mathbb{C}, \#\{i \mid \mu_i = 0\} \neq 1 \right\}$$
Tensors with monic rank \( \leq 2 \)

Claim: The set of sums of two monic tensors with rank 1 is

\[
\mathbb{C}^3 \cdot \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 & \mu_1 \\ 0 & \mu_2 & 0 \end{pmatrix} \bigg| \begin{array}{c} \mu_1, \mu_2, \mu_3 \in \mathbb{C}, \\ \#\{i \mid \mu_i = 0\} \neq 1 \end{array} \right\}
\]

Proof:

\[
\begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 & \mu_1 \\ 0 & \mu_2 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & b & c & bc \\ a & ab & ac & abc \end{pmatrix} + \begin{pmatrix} 1 & -b & -c & bc \\ -a & ab & ac & -abc \end{pmatrix}
\]
Tensors with monic rank \( \leq 2 \)

Claim: The set of sums of two monic tensors with rank 1 is

\[
\mathbb{C}^3 \cdot \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \mid \begin{array}{c} \mu_1, \mu_2, \mu_3 \in \mathbb{C}, \\ \#\{i \mid \mu_i = 0\} \neq 1 \end{array} \right\}
\]

Proof:

\[
\begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & b & c & bc \\ a & ab & ac & abc \end{pmatrix} + \begin{pmatrix} 1 & -b & -c & bc \\ -a & ab & ac & -abc \end{pmatrix}
\]

\[
= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2ab & 2ac \\ 0 & 2ab & 0 \end{pmatrix}
\]
Tensors with monic rank $\leq 2$

**Claim:** The set of sums of two monic tensors with rank 1 is

$$\mathbb{C}^3 \cdot \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 & \mu_1 \\ 0 & \mu_2 & 0 \end{pmatrix} \mid \mu_1, \mu_2, \mu_3 \in \mathbb{C}, \# \{ i \mid \mu_i = 0 \} \neq 1 \right\}$$

**Proof:**

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 & \mu_1 \end{pmatrix} = \begin{pmatrix} 1 & b & c & bc \\ a & ab & ac & abc \end{pmatrix} + \begin{pmatrix} 1 & -b & -c & bc \\ -a & ab & ac & -abc \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2ab & 2ac & 0 \end{pmatrix}$$

**Idea:** Write every tensor with leading coefficient 3 as

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 & \mu_1 \end{pmatrix} + \begin{pmatrix} 1 & b & c & bc \\ a & ab & ac & abc \end{pmatrix}$$

with $\mu_1, \mu_2, \mu_3 \in \mathbb{C} \setminus \{0\}$ and $a, b, c \in \mathbb{C}$. 
Tensors with monic rank \( \leq 3 \)

Start with a tensor with leading coefficient 3 in standard form.

\[
\begin{pmatrix}
3 & 0 & 0 \\
0 & \mu_3 & \mu_2 \\
0 & \mu_2 & \lambda
\end{pmatrix}
\]
Tensors with monic rank $\leq 3$

Start with a tensor with leading coefficient 3 in standard form.

$$\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right) \cdot \begin{pmatrix} 3 & 0 & 0 \\ 0 & \mu_3 & \mu_2 \\ 0 & \mu_1 & \lambda \end{pmatrix}$$
Tensors with monic rank $\leq 3$

Start with a tensor with leading coefficient 3 in standard form.

$\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right) \cdot \begin{pmatrix} 3 & 0 & 0 \\ 0 & \mu_3 & \mu_1 \\ \mu_2 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & b & c & bc \\ a & ab & ac & abc \end{pmatrix}$
Start with a tensor with leading coefficient 3 in standard form.

\[
\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right) \cdot \begin{pmatrix} 3 & 0 & 0 \\ 0 & \mu_3 & \mu_1 \\ \mu_2 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & b & c \\ a & ab & ac \\ ab & abc \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \mu_3 - \frac{2}{3}ab & \mu_2 - \frac{2}{3}ac \\ 0 & \mu_3 - \frac{2}{3}ab & \mu_2 - \frac{2}{3}ac \end{pmatrix} \begin{pmatrix} \mu_1 - \frac{2}{3}bc \\ \mu_1 - \frac{2}{3}bc \\ \mu_1 - \frac{2}{3}bc \end{pmatrix} - \frac{8}{9}abc
\]
Tensors with monic rank $\leq 3$

Start with a tensor with leading coefficient 3 in standard form.

$\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right) \cdot \left(\begin{array}{cc|cc} 3 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right) - \left(\begin{array}{cc|cc} 1 & b & c & bc \\ a & ab & ac & abc \end{array}\right) =$

$\left(\begin{array}{cc|cc} 2 & 0 & 0 & \mu_1 - \frac{2}{3}bc \\ 0 & \mu_3 - \frac{2}{3}ab & \mu_2 - \frac{2}{3}ac & \lambda + \frac{1}{3}(a\mu_1 + b\mu_2 + c\mu_3) - \frac{8}{9}abc \end{array}\right)$

Want:

- $\lambda + \frac{1}{3}(a\mu_1 + b\mu_2 + c\mu_3) - \frac{8}{9}abc = 0$
- $\mu_1 - \frac{2}{3}bc \neq 0, \mu_2 - \frac{2}{3}ac \neq 0$ and $\mu_3 - \frac{2}{3}ab \neq 0$
## Tensors with monic rank \( \leq 3 \)

Start with a tensor with leading coefficient 3 in standard form.

\[
\left( \frac{1}{3} a, \frac{1}{3} b, \frac{1}{3} c \right) \cdot \left( \begin{array}{c|cc}
3 & 0 & \mu_1 \\
0 & \mu_3 & \mu_2 \\
\end{array} \right) - \left( \begin{array}{c|cc}
1 & b & c \\
1 & a b & ac \\
\end{array} \right) = \\
\left( \begin{array}{c|cc}
2 & 0 & 0 \\
0 & \mu_3 - \frac{2}{3} ab & \mu_2 - \frac{2}{3} ac \\
\end{array} \right) \left( \begin{array}{c|cc}
\mu_1 - \frac{2}{3} bc & & \\
\mu_1 - \frac{2}{3} bc & \mu_1 - \frac{2}{3} bc & \\
\lambda + \frac{1}{3}(a\mu_1 + b\mu_2 + c\mu_3) - \frac{8}{9} abc & & \\
\end{array} \right)
\]

**Want:**

- \( \lambda + \frac{1}{3}(a\mu_1 + b\mu_2 + c\mu_3) - \frac{8}{9} abc = 0 \)
- \( \mu_1 - \frac{2}{3} bc \neq 0, \mu_2 - \frac{2}{3} ac \neq 0 \) and \( \mu_3 - \frac{2}{3} ab \neq 0 \)

This is doable unless \( \lambda = \mu_1 = \mu_2 = \mu_3 = 0 \) (and that case is easy).
Maximal rank vs maximal monic rank

Theorem

- For an $n \times m$ matrix, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = \min(n, m) \]
- For a symmetric $n \times n$ matrix, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = n \]
- For a trace-zero $n \times n$ matrix, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = n \]
- For a $2 \times 2 \times 2$ tensor, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = 3 \]
Maximal rank vs maximal monic rank

Theorem

- For an $n \times m$ matrix, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = \min(n, m) \]
- For a symmetric $n \times n$ matrix, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = n \]
- For a trace-zero $n \times n$ matrix, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = n \]
- For a $2 \times 2 \times 2$ tensor, we have
  \[ \text{maximal rank} = \text{maximal monic rank} = 3 \]

Assume that $h$ is a highest weight vector.

Question: Are the maximal rank and maximal monic rank always equal?
References

