Strength and polynomial functors

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The rank of infinite-by-infinite matrices

Definition: The rank of an $\mathbb{N} \times \mathbb{N}$ matrix $A$ is

$$\operatorname{rk}(A) := \sup\{\operatorname{rk}(B) \mid \text{finite submatrices } B \text{ of } A \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\}$$

Lemma

$A \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ has rank $\leq k \iff A = \sum_{i=1}^{k} v_i w_i^T$ with $v_i, w_i \in \mathbb{C}^\mathbb{N}$
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**Example/Theorem**

An $\mathbb{N} \times \mathbb{N}$ matrix $A$ has rank $\infty \iff \text{GL}_\infty \cdot A \cdot \text{GL}_\infty = \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$
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Lemma
\( A \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \) has rank \( \leq k \) \( \iff \) \( A = \sum_{i=1}^{k} v_i w_i^T \) with \( v_i, w_i \in \mathbb{C}^{\mathbb{N}} \)

Example/Theorem
An \( \mathbb{N} \times \mathbb{N} \) matrix \( A \) has rank \( \infty \) \( \iff \) \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty = \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \)

Proof. An equation on \( \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \) uses only finitely many rows and columns. So non-zero equations on \( \text{GL}_\infty \cdot A \cdot \text{GL}_\infty \) give rank constraints on \( A \). \( \square \)

Fact: An \( n \times m \) matrix \( A \) has rank \( \min(n, m) \) \( \iff \) \( \text{GL}_n \cdot A \cdot \text{GL}_m = \mathbb{C}^{n \times m} \)
**Other Examples/Theorems**

**Definition:** The rank of a tuple of $\mathbb{N} \times \mathbb{N}$ matrices $A_1, \ldots, A_k$ is

$$
\text{rk}(A_1, \ldots, A_k) := \inf \{ \text{rk}(\lambda_1 A_1 + \cdots + \lambda_k A_k) \mid (\lambda_1 : \cdots : \lambda_k) \in \mathbb{P}^{k-1} \}
$$

**Example/Theorem** (Draisma-Eggermont)

\[
\text{rk}(A_1, \ldots, A_k) = \infty \iff \overline{\text{GL}_\infty \cdot (A_1, \ldots, A_k) \cdot \text{GL}_\infty} = (\mathbb{C}^{\mathbb{N} \times \mathbb{N}})^k
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**Definition:** The q-rank of a series

\[
f = a_{111} x_1^3 + a_{112} x_1^2 x_2 + \cdots + a_{ijk} x_i x_j x_k + \ldots
\]

is the minimal \( k \leq \infty \) such that

\[
f = \ell_1 q_1 + \cdots + \ell_k q_k \text{ with } \deg(\ell_i) = 1.
\]

**Example/Theorem (Derksen-Eggermont-Snowden)**

\[
\text{qrk}(f) = \infty \iff \text{GL}_\infty \cdot f = \lim_n \mathbb{C}[x_1, \ldots, x_n](3)
\]
Other Examples/Theorems

Take $d \geq 2$.

**Definition** (Ananyan-Hochster)
The strength of a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]_{(d)}$ is the minimal $k$ such that

$$f = g_1 h_1 + \cdots + g_k h_k$$

with $g_1, \ldots, g_k, h_1, \ldots, h_k \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous of degree $< d$. 
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**Example/Theorem** (B-Draisma-Eggermont, Kazhdan-Ziegler)
For every \( n \), let \( X_n \subseteq \mathbb{C}[x_1, \ldots, x_n]_d \) be a closed subset such that:

\((\ast)\) We have \( f \circ \ell \in X_m \) for all \( f \in X_n \) and all linear maps \( \ell: \mathbb{C}^m \to \mathbb{C}^n \).
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Then either $X_n = \mathbb{C}[x_1, \ldots, x_n]_d$ for all $n \geq 0$. 

Remark: This version implies the infinite version using Lang's theorem.
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Then either \( X_n = \mathbb{C}[x_1, \ldots, x_n]_{(d)} \) for all \( n \geq 0 \) or there is a \( k < \infty \) such that \( \text{str}(f) \leq k \) for all \( f \in X_n \) and \( n \geq 0 \).
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**Remark**: This version implies the infinite version using Lang’s theorem.
Polynomial functors

$\mathbf{Vec} =$ category of finite-dimensional vector spaces over $\mathbb{C}$.

**Definition**

A polynomial functor $P$ assigns to $V \in \mathbf{Vec}$ a $P(V) \in \mathbf{Vec}$ and to $(V, W) \in \mathbf{Vec}^2$ a polynomial map $\hom_{\mathbb{C}}(V, W) \to \hom_{\mathbb{C}}(P(V), P(W))$ such that $P(\text{id}_V) = \text{id}_{P(V)}$ for all $V \in \mathbf{Vec}$ and $P(\varphi \circ \psi) = P(\varphi) \circ P(\psi)$ for all linear maps $\psi : V \to W$ and $\varphi : W \to U$.

**Examples**

- Constants: $V \mapsto U$ for $U \in \mathbf{Vec}$ fixed.
- Linear functors: $V \mapsto U \otimes V$ for $U \in \mathbf{Vec}$ fixed.
- Matrices: $V \mapsto V \otimes V$
- Polynomials: $V \mapsto S^d V$

**Remark:** The class of polynomial functors is closed under direct sums, tensor products, quotients and subfunctors. Polynomial functors have a degree. (This can be infinite, but we don’t consider such poly functors.)
Polynomial transformations and Closed subsets of polynomial functors

**Definition**

Let $P, Q$ be polynomial functors. A polynomial transformation $\alpha : Q \rightarrow P$ is a family $(\alpha_V : Q(V) \rightarrow P(V))_{V \in \text{Vec}}$ of polynomial maps such that

$$
\begin{array}{c}
Q(V) \xrightarrow{\alpha_V} P(V) \\
| \quad | \\
Q(\ell) \quad P(\ell) \\
| \quad | \\
Q(W) \xrightarrow{\alpha_W} P(W)
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$$

commutes for all linear maps $\ell : V \rightarrow W$. 
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**Definition**

A closed subset $X \subseteq P$ of a polynomial functor assigns to each $V \in \text{Vec}$ a closed subset $X(V) \subseteq P(V)$ such that $p(\varphi)(X(V)) \subseteq X(W)$ for all linear maps $\ell : V \to W$. 
The dichotomy

Let $P, Q$ be polynomial functors. Write $Q < P$ when $Q_{(d)}$ is a quotient of $P_{(d)}$ where $d$ is maximal with $Q_{(d)} \not\cong P_{(d)}$. 
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**Theorem** (B-Draisma-Eggermont-Snowden)

Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are $Q_1, \ldots, Q_k < P$ and $\alpha_i : Q_i \to P$ such that $X \subseteq \bigcup_i \text{im}(\alpha_i)$. 
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**Examples**
- \{matrices of rank $\leq k\} = \{v_1 w_1^T + \cdots + v_k w_k^T \mid v_i, w_i \text{ vectors}\}$
- \{degree $d$ polynomials that are zero on a codim $k$ subspace\} = \{\ell_1 g_1 + \cdots + \ell_k g_k \mid \text{deg}(\ell_i) = 1, \text{deg}(g_i) = d - 1\}
Consequences

- All the previous Examples/Theorems
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- **Theorem** (Draisma)
  Every descending chain $P \supseteq X_1 \supseteq X_2 \supseteq \ldots$ of closed subsets stabilizes.
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  **Proof.** Using induction on $P$:
  Take $Q_1, \ldots, Q_k < P$ and $\alpha_i : Q_i \to P$ such that $X_1 \subseteq \bigcup_i \text{im}(\alpha_i)$ and pull back the chain of closed subsets along each $\alpha_i$. The resulting chains all have to stabilize.
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- **Theorem** (B-Draisma-Eggermont-Snowden)
  The map $\alpha \mapsto \overline{\text{im}(\alpha)}$ is a surjection from
  \{polynomial transformations into $P$\} to
  \{closures of $GL_\infty$-orbits in $\lim_n P(\mathbb{C}^n)$\}. 
References


