ED Degrees of Orthogonally Invariant Varieties

Arthur Bik
Mathematical Institute
University of Bern

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ED degree of a variety

Fix a finite-dimensional complex vector space $V$, a non-degenerate symmetric bilinear form on $V$, a closed algebraic subvariety $X$ of $V$ (+ conditions).
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Then for a sufficiently general $v \in V$ the positive number

$$\# \left\{ x \in X^{\reg} \left| v - x \perp T_x X \right. \right\}$$

is independent of $v$ and is called the ED degree of $X$ in $V$. 

ED degree of a variety
Example: unit circle

\[ x^2 + y^2 = 1 \]
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Orthogonally invariant matrix varieties

The group $O(n) \times O(m)$ acts on the space $\mathbb{C}^{n \times m}$ of $n \times m$ matrices. The bilinear form

$$ (A, B) \mapsto \text{Tr}(AB^T) $$

is invariant.
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**Theorem (Drusvyatskiy, Lee, Ottaviani, Thomas, 2016)**

Let $X$ be the closure in $\mathbb{C}^{n \times m}$ of a stable real subvariety of $\mathbb{R}^{n \times m}$ with smooth points and let $X_0$ be the subset of $X$ of diagonal matrices. Then the ED degree of $X$ in $\mathbb{C}^{n \times m}$ equals the ED degree of $X_0$ in the subspace of $\mathbb{C}^{n \times m}$ of all diagonal matrices.
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Let \( X \) be the closure in \( \mathbb{C}^{n \times m} \) of a stable real subvariety of \( \mathbb{R}^{n \times m} \) with smooth points and let \( X_0 \) be the subset of \( X \) of diagonal matrices. Then the ED degree of \( X \) in \( \mathbb{C}^{n \times m} \) equals the ED degree of \( X_0 \) in the subspace of \( \mathbb{C}^{n \times m} \) of all diagonal matrices.

Observations:
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Observations:

**(1)** $O(n)X_0O(m)$ is dense in $X$. (Singular Value Decomposition)
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Observations:

1. $O(n)X_0O(m)$ is dense in $X$. (Singular Value Decomposition)
2. For $D \in \mathbb{C}^{n \times m}$ a sufficiently general diagonal matrix, we have
   $$\mathbb{C}^{n \times m} = \{\text{diagonal matrices}\} \oplus T_D (O(n)DO(m)).$$
Orthogonally invariant varieties

Let $V$ be an orthogonal representation of an algebraic group $G$. Let $X$ be a $G$-stable closed subvariety of $V$ (+ conditions).
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Let $V$ be an orthogonal representation of an algebraic group $G$. Let $X$ be a $G$-stable closed subvariety of $V$ (+ conditions).

**Theorem (B, Draisma, 2017)**

Let $V_0 \subseteq V$ be a subspace and set $X_0 := X \cap V_0$. Assume that $GX_0$ is dense in $X$ and that

$$V = V_0 \oplus T_{v_0} Gv_0$$

for sufficiently general $v_0 \in V_0$. Then the ED degree of $X$ in $V$ equals the ED degree of $X_0$ in $V_0$. 
Sketch of proof

Let \( v \in \mathcal{V} \) and \( v_0 \in \mathcal{V}_0 \) be sufficiently general. We want:

\[
\#
\]

Lemma.

\( \mathcal{G}_0 \) is dense in \( \mathcal{V} \).

\( \Rightarrow \) may assume \( v = \tilde{\tilde{v}}_0 \).

Lemma.

\( g \) maps critical points of \( u \) to critical points of \( gu \) one-to-one.

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Critical points of \( v_0 \) for \( X \) and \( X_0 \) are same.
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Let \( v \in V \) and \( v_0 \in V_0 \) be sufficiently general. We want:

\[
\# \left\{ x \in X_{\text{reg}} \mid v - x \perp T_x X \right\} = \# \left\{ x \in X_{0 \text{reg}}^{\text{reg}} \mid v_0 - x \perp T_x X_0 \right\}
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**Lemma.** $GV_0$ is dense in $V$. 
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$$V = V_0 \oplus T_{v_0} G v_0$$

*for sufficiently general $v_0 \in V_0$. Then the ED degree of $X$ in $V$ equals the ED degree of $X_0$ in $V_0$.***
Let $n > 0$ be an integer. Take $G = \text{GL}_n$ acting on

$$V = \{(A, B) \in (\mathbb{C}^{n \times n})^2 | A = A^T, B = B^T\}$$

by $g \cdot (A, B) = (gAg^T, g^{-T}Bg^{-1})$. 
Example (Jiri Dadok)

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$$((A, B), (C, D)) \mapsto \text{Tr}(AD + BC)$$

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is invariant. Take \( V_0 = \{(D, D) | D \in \mathbb{C}^{n \times n} \text{ diagonal}\} \). Then

\[
V = V_0 \oplus T_{(D, D)}G(D, D)
\]

for all invertible \( D = \text{diag}(d_1, \ldots, d_n) \) with \( d_i^2 \neq d_j^2 \) for \( i \neq j \).
Let $G$ be reductive. Let $K$ be a maximal compact subgroup of $G$ and let $V_R$ a real representation of $K$ whose complexification is $V$.

Theorem (B, Draisma, 2017)

The following are equivalent:

1. $V$ has a subspace $V_0$ such that $V_0$ is $T$-invariant for sufficiently general $v_0 \in V_0$.

2. $V$ is a stable polar representation.

3. $V_R$ is a polar representation.

Dadok classified irreducible polar representations of compact Lie groups.
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The following are equivalent:

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Polar representations

Definition
A complex representation $V$ of an reductive algebraic group $G$ is stable polar if there is a vector $v \in V$, whose orbit is maximal-dimensional and closed, such that the subspace

$$\{ x \in V | T_x G x \subseteq T_v G v \}$$

has dimension $\dim(V/G)$. 

Definition
A real representation $V$ of a compact Lie group $K$ is polar if there is a vector $v \in V$, whose orbit is maximal-dimensional, such that for all $u \in p T_v G v$, we have $T_u K u \subseteq T_v G v$. 

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**Definition**
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# Classification

## Complexification of Dadok’s list:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ semisimple</td>
<td>$g$</td>
</tr>
<tr>
<td>$O(n)$</td>
<td>$\mathbb{C}^n$</td>
</tr>
<tr>
<td>$O(n)$</td>
<td>$\text{Sym}^2(\mathbb{C}^n)$</td>
</tr>
<tr>
<td>$O(n) \times O(m)$</td>
<td>$\mathbb{C}^{n \times m}$</td>
</tr>
<tr>
<td>$\text{Sp}(n)$</td>
<td>$\Lambda^2(\mathbb{C}^{2n})$</td>
</tr>
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<td>$\text{Sp}(n) \times \text{Sp}(m)$</td>
<td>$\mathbb{C}^{2n \times 2m}$</td>
</tr>
<tr>
<td>$\text{SL}(V)$</td>
<td>$V \oplus V^*$</td>
</tr>
<tr>
<td>$\text{GL}(V)$</td>
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</tr>
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</tr>
<tr>
<td>$\text{SL}_2$</td>
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<tr>
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Thank you for your attention!
References


