# Hamiltonian and Nilpotent Residuated Lattices

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In 1983, Reilly studied the class of nilpotent $\ell$-groups and its connection with the class of Hamiltonian $\ell$-groups. Nilpotent $\ell$-groups are those $\ell$-groups with a central series of finite length (i.e., whose group reduct is a nilpotent group), and Hamiltonian $\ell$-groups are those for which every convex subalgebra is normal. The link between these two classes includes but is not limited to the fact that they can be seen as generalizations of the variety of Abelian $\ell$-groups. Kopytov showed in [6, Corollary 2] that every nilpotent $\ell$-group is Hamiltonian (proved independently by Reilly in [11, Theorem 2.4]) and, in 1984, asked the question whether every variety of Hamiltonian $\ell$-groups is contained in the variety generated by all nilpotent $\ell$-groups. The problem remained open for two decades, until it was answered negatively in [1, Theorem A].

In this work, we consider prelinear cancellative residuated lattices as a natural generalization of $\ell$-groups. Here the term prelinear denotes those residuated lattices that satisfy the following two laws: $((x\backslash y) \wedge \mathrm{e}) \vee ((y\backslash x) \wedge \mathrm{e}) \approx \mathrm{e}$ and $((y/x) \wedge \mathrm{e}) \vee ((x/y) \wedge \mathrm{e}) \approx \mathrm{e}$. We exploit well-known results about nilpotent and Hamiltonian $\ell$-groups to argue for similar results about nilpotent and Hamiltonian prelinear cancellative residuated lattices.

Hamiltonian residuated lattices have been studied in [2], where Hamiltonian varieties of e-cyclic residuated lattices were characterized. It was proved that that there is no largest variety of Hamiltonian e-cyclic residuated lattices ([2, Theorem 6.3]). Nonetheless, the fact that there exists a largest Hamiltonian variety of $\ell$-groups (namely, the variety of weakly Abelian $\ell$-groups [11, Theorem 2.2]) rises the question whether this is also the case for varieties of Hamiltonian prelinear cancellative residuated lattices. The answer to this question is positive.

**Theorem A.** There exists a largest variety of Hamiltonian prelinear cancellative residuated lattices. More precisely, a variety $\mathcal{V}$ of prelinear cancellative residuated lattices is Hamiltonian if and only if $\mathcal{V}$ satisfies the equations $(x \wedge \mathrm{e})^2 \leq \lambda_u(x)$ and $(x \wedge \mathrm{e})^2 \leq \rho_v(x)$, where $\lambda_u(x) = (u\backslash xu) \wedge \mathrm{e}$ and $\rho_v(x) = (vx/v) \wedge \mathrm{e}$.

In [9], Neumann and Taylor identified those cancellative semigroups that can be embedded into nilpotent groups of class $c \in \mathbb{N} - \{0\}$ (cf. [7]), that is, those groups whose shortest central series has length $c$. They are defined by a semigroup equation $L_c\colon q_c(x, y, \bar{z}) = q_c(y, x, \bar{z})$, where $\bar{z}$ abbreviates a sequence of variables $z_1, z_2, \dots,$ and $q_c(x, y, \bar{z})$ is inductively defined as follows: $q_1(x, y, \bar{z}) = xy$, and $q_{c+1}(x, y, \bar{z}) = q_c(x, y, \bar{z}) z_c q_c(y, x, \bar{z})$. The semigroup equation $L_c$ is proved to equationally characterize the variety of nilpotent groups of class $c$ (see [9, Corollary 1]). Hence, we define nilpotent class-$c$ residuated lattices as those whose underlying semigroup satisfies the semigroup equation $L_c$.

In [8, Corollary 5.2], the category of commutative cancellative residuated lattices is proved to be equivalent to the category of Abelian $\ell$-groups endowed with a conucleus whose image generates the underlying group of the $\ell$-group. In this work, this result is extended to nilpotent cancellative residuated lattices and nilpotent $\ell$-groups. More precisely, let $\mathcal{LG}_{cn}$ be the category with objects $\langle \mathbf{G}, \sigma \rangle$ consisting of an $\ell$-group $\mathbf{G}$ augmented with a conucleus $\sigma$ such that the group reduct of $\mathbf{G}$ is the group of left quotients of the underlying monoid of $\sigma(\mathbf{G})$, and with morphisms given by $\ell$-groups homomorphisms commuting with the conuclei. Let $\mathcal{NC}an\mathcal{RL}$ be the category of nilpotent cancellative residuated lattices and residuated lattice homomorphisms, and let $\mathcal{N}_{cn}$ the full subcategory of $\mathcal{LG}_{cn}$ consisting of objects whose first component is a nilpotent $\ell$-group.

**Theorem B.** The categories $\mathcal{NC}an\mathcal{RL}$ and $\mathcal{N}_{cn}$ are equivalent. The description of the functors $\Omega$ and $\Omega^{-1}$ implementing the equivalence is given in [8, Definition 4.6] and [8, Definition 4.7].

The fact that nilpotent $\ell$-groups are semilinear[1] ([6, 4]), that is, they are subdirect products of totally ordered algebras, allows us to make use of Theorem B and obtain the following result.

**Theorem C.** Nilpotent prelinear cancellative residuated lattices are semilinear.

The proof of Theorem C makes use of the fact that, in the presence of prelinearity, semilinear residuated lattices can be defined by the quasi-equation ([2, Theorem 5.6]): $x \vee y \approx \mathrm{e} \Longrightarrow \lambda_u(x) \vee \rho_v(y) \approx \mathrm{e}$. It also exploits an interesting reformulation of prelinearity, holding when the following equation is in place: $\mathrm{e} \wedge (y \vee z) \approx (\mathrm{e} \wedge y) \vee (\mathrm{e} \wedge z)$.

We make use of [8, Corollary 5.5], together with Theorem B and Theorem C, to conclude that nilpotent prelinear cancellative residuated lattices are Hamiltonian, generalizing the same result for nilpotent $\ell$-groups [6, 11].

**Theorem D.** Nilpotent prelinear cancellative residuated lattices are Hamiltonian.

Finally, relying on [10, Theorem 2.2] and on [3, Theorem 4.3], we prove the following result.

**Theorem E.** The amalgamation property fails for the variety of nilpotent class-$c$ prelinear cancellative residuated lattices.

This result is to be expected, based on the fact that the varieties of nilpotent $\ell$-groups are proved to fail the amalgamation property ([10]). As a consequence of this and Theorem D, the variety of nilpotent class-$c$ prelinear cancellative residuated lattices fails the deductive interpolation property, given that every Hamiltonian variety satisfies the congruence extension property ([5, Corollary 11]).


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