Theorems of the Alternative in Substructural Logics

Almudena Colacito

Mathematical Institute, University of Bern, Switzerland
almudena.colacito@math.unibe.ch

Abelian lattice-ordered groups (briefly, abelian ℓ-groups) are defined as abelian groups equipped with a lattice order that is compatible with the group operation, and play a fundamental role in the study of substructural and many-valued logics (see, e.g., [2]). Let $\mathcal{A}b$ denote the variety of abelian groups over the signature $\{+, -, 0\}$, and let $\mathcal{A}$ denote the variety of abelian ℓ-groups. It is known that:

Theorem 1. *Given finitely many group terms $t_1, \ldots, t_n$, the following statements are equivalent:*

(a) $\mathcal{A} \models 0 \leq t_1 \vee \cdots \vee t_n$.

(b) $\mathcal{A}b \models 0 \approx \lambda_1 t_1 + \cdots + \lambda_n t_n$, *for some* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ *not all* 0.

A proof of Theorem 1 can be found in [3] and provides a completeness theorem for a one-sided variant of the hypersequent calculus for abelian ℓ-groups introduced in [7]. Using the fact that $\mathcal{A}b$ is generated as a variety by the group of integers and that $\mathcal{A}$ is generated by the ℓ-group of integers with the standard order, the equivalence between (a) and (b) can be seen as a variant of the following theorem of the alternative:

Gordan's Lemma. *Given a matrix $A \in \mathbb{Z}^{m \times n}$, exactly one of the following statements holds:*

(i) *There exists $y \in \mathbb{Z}^m$ such that $y^T A < 0$.*

(ii) *There exists $z \in \mathbb{N}^n \setminus \{0\}$ such that $Az = 0$.*

This is an integer-valued version of Gordan's Lemma, a fundamental theorem of the alternative (also called 'of Farkas type') in linear algebra. Theorems of the alternative may be understood as duality principles stating that either one or another linear system has a solution over the real numbers, but not both (see, e.g., [5]).

The aim of this work is to inquire how far Theorem 1 – and more generally, theorems of the alternative – can be extended in the setting of substructural logic. A first possibility is to consider more general varieties of ℓ-groups, and arises from the observation that part (b) of Theorem 1 is equivalent to the fact that the reduced group terms $\bar{t}_1, \ldots, \bar{t}_n$ do not extend to a total order of the free abelian group. This approach, pursued in [3, 4], results in correspondences between validity of equations in classes of ℓ-groups and extending partial (right) orders to total (right) orders of free groups. Here, we approach the problem from a different perspective, aiming for a characterization of those classes of commutative residuated po-monoids with an involution and a fixed point that satisfy similar duality results.

An *involutive commutative residuated po-monoid* (briefly, involutive CRPM) is a structure $\langle \mathbf{A}, \leq \rangle$, with $\mathbf{A} = \langle A, \cdot, \to, 1, 0 \rangle$, where $\langle A, \cdot, 1 \rangle$ is a commutative monoid, the relation $\leq$ is a partial order on A compatible with the monoid operation (in this case, we say that $\langle A, \cdot, \to, 1, \leq \rangle$ is a *commutative po-monoid*), the operation $\cdot$ is residuated with residual $\to$, and the constant 0 is a dualizing element of A. It is convenient for our approach to consider the term-equivalent structures $\langle A, +, -, 0, \leq \rangle$ over the additive signature $\{+, -, 0\}$, where $\langle A, +, 0, \leq \rangle$ is a commutative po-monoid, the unary operation $-$ is an involution on $\langle A, \leq \rangle$, and $a \cdot b \leq c \iff a \leq b \to c$, for all $a, b, c \in A$, where $a \cdot b$ stands for $-((-a) + (-b))$ and $b \to c$ stands for $-b + c$. We say that an involutive CRPM is *fixed point* if also $-0 = 0$. Note that the class of involutive CRPMs provides a semantics for the multiplicative fragment of linear logic (see [1]), and that every partially ordered abelian group is a fixed point involutive CRPM where $a \cdot b = a + b$.

We consider any class $\mathcal{K}$ of structures defined relative to the class of fixed point involutive CRPMs by a set of inequations, and the corresponding variety $\mathcal{K}^\ell$ generated by the totally ordered members of $\mathcal{K}$ equipped with lattice operations $\wedge$ and $\vee$. Note that $\mathcal{K}$ is a semantics for an axiomatic extension of the multiplicative fragment of linear logic, and $\mathcal{K}^\ell$ is a semantics for the corresponding axiomatic extension of involutive uninorm logic (see [6]). We say that a term t is a group term if it contains only variables and operations from the signature $\{+, -, 0\}$. We define also the following shortcuts: $0a := 0$, $(n+1)a := na + a$, $a^0 := -0$, and $a^{(n+1)} := a^n \cdot a$, for $n \in \mathbb{N}$.

We say that a class $\mathcal{K}$ of fixed point involutive CRPMs admits a *theorem of the alternative* if the following holds:

Property 1. *Given finitely many group terms $t_1, \ldots, t_n$, the following statements are equivalent:*

(a) $\mathcal{K}^\ell \models 0 \leq t_1 \vee \cdots \vee t_n$.

(b) $\mathcal{K} \models 0 \leq \lambda_1 t_1 + \cdots + \lambda_n t_n$, *for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all* 0.

Note that Theorem 1 becomes now a particular instance of Property 1 if we take $\mathcal{K}$ to be the class of partially ordered abelian groups and $\mathcal{K}^\ell$ to be the variety $\mathcal{A}$ of abelian ℓ-groups. Observe also that Property 1 can be used to obtain a hypersequent calculus for $\mathcal{K}^\ell$ from a given sequent calculus for $\mathcal{K}$.

Given a set $\Sigma \cup \{s \leq t\}$ of inequations of group terms, we write $\Sigma \models_{\mathcal{K}} s \leq t$ to denote that for any $\langle \mathbf{A}, \leq \rangle \in \mathcal{K}$ and valuation φ on $\mathbf{A}$, if $\varphi(u) \leq \varphi(v)$ for all $u \leq v \in \Sigma$, then also $\varphi(s) \leq \varphi(t)$. Similarly, by writing $\Sigma \models_{\mathcal{K}^\ell} s \leq t$ we mean that for any $\mathbf{A} \in \mathcal{K}^\ell$ and valuation φ on $\mathbf{A}$ such that $\varphi(u) \wedge \varphi(v) = \varphi(u)$ for all $u \leq v \in \Sigma$, also $\varphi(s) \wedge \varphi(t) = \varphi(s)$. We say that $\mathcal{K}^\ell$ is a *weak conservative extension* of $\mathcal{K}$ if for any finite set $S \cup \{t\}$ of group terms:

$$\{0 \leq s \mid s \in S\} \models_{\mathcal{K}^\ell} 0 \leq t \iff \{0 \leq s \mid s \in S\} \models_{\mathcal{K}} 0 \leq \lambda t, \text{ for some } \lambda \in \mathbb{N}^+.$$

We obtain the following characterization:

Theorem 2. *The following conditions are equivalent:*

1. *The class $\mathcal{K}$ admits a theorem of the alternative.*
2. *For each $n \in \mathbb{N}$, there exist $\kappa_1, \kappa_2 \in \mathbb{N}$ such that $\mathcal{K} \models (nx)^{\kappa_1} \leq \kappa_2 x^n$.*
3. *The variety $\mathcal{K}^\ell$ is a weak conservative extension of $\mathcal{K}$.*

The proof of Theorem 2 heavily relies on the facts that the logic order-algebraized by $\mathcal{K}$ satisfies a local deduction theorem, and that every term of $\mathcal{K}^\ell$ is equivalent to meets of joins of group terms.

Apart from partially ordered abelian groups, the most striking example satisfying the equivalent conditions of Theorem 2 is the class of idempotent fixed point involutive CRPMs, which provides a semantics for the multiplicative fragment of involutive uninorm mingle logic. The class $\mathcal{K}^\ell$ generated by totally ordered idempotent fixed point involutive CRPMs is the variety of odd Sugihara monoids.

References

[1] A. Avron. The semantics and proof theory of linear logic. *Theoretical Computer Science*, 57(2-3):161–184, 1988.

[2] R. L. O. Cignoli, I. M. L. D'Ottaviano and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*. Kluwer, 1999.

[3] A. Colacito and G. Metcalfe. Ordered groups and proof theory. *Proceedings of WoLLIC 2017*, Vol. 10388 of LNCS, 80–91. Springer, 2017.

[4] A. Colacito and G. Metcalfe. Ordering groups and validity in lattice-ordered groups. *Submitted, December 2017*.

[5] G. B. Dantzig. *Linear Programming and Extensions*. Princeton University Press, 1963.

[6] G. Metcalfe and F. Montagna. Substructural fuzzy logics. *Journal of Symbolic Logic*, 72(3):834–864, 2007.

[7] G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for Abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.