Theorems of Alternatives: An Application to Densifiability

Almudena Colacito[1], Nikolaos Galatos[2], and George Metcalfe[1]

[1] Mathematical Institute, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland
{almudena.colacito,george.metcalfe}@math.unibe.ch
[2] Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208, USA
ngalatos@du.edu

1 Introduction

A variety $\mathcal{V}$ of semilinear residuated lattices is called *densifiable* if it is generated as a quasivariety by its dense chains, or, equivalently, each chain in $\mathcal{V}$ embeds into a dense chain in $\mathcal{V}$ (see [1, 11, 6, 3]). Establishing that some variety is densifiable is a fundamental problem of mathematical fuzzy logic, corresponding to a key intermediate step in proving that a given axiom system is "standard complete": that is, complete with respect to a class of algebras with lattice reduct $[0, 1]$ (see, e.g., [8, 9, 2]).

Densifiability may be established using representation theorems or by providing explicit embeddings of countable chains into dense countable chains of the variety. The latter approach, introduced in [8], has been used to establish densifiability for varieties of integral semilinear residuated lattices, but can be difficult to apply in the non-integral setting. An alternative proof-theoretic method, used in [9, 2] to establish densifiability for a range of integral and non-integral varieties, circumvents the need to give explicit embeddings. Instead, the elimination of a certain density rule for a hypersequent calculus is used to prove that the variety satisfies a property that guarantees densifiability. Remarkably, this method has also been reinterpreted algebraically to obtain explicit embeddings of chains into dense chains [6, 1].

The methods described above are suitable for varieties of semilinear residuated lattices that admit either a useful representation theorem (e.g., via ordered groups) or an analytic hypersequent calculus. In this work, we introduce a method for establishing densifiability for varieties that may not satisfy either of these conditions, but admit instead a "theorem of alternatives" relating validity of equations in the variety to validity of equations in its residuated monoid reduct. Although the scope of this method is fairly narrow — applying so far only to varieties of involutive commutative semilinear residuated lattices — it yields both new and familiar (e.g., abelian ℓ-groups and odd Sugihara monoids) examples of densifiable varieties, and provides perhaps a first step towards addressing the open standard completeness problem for involutive uninorm logic posed in [9].[1]

2 Theorems of Alternatives

Theorems of alternatives can be understood as duality principles stating that either one or another linear system has a solution over the real numbers, but not both (see, e.g., [5]). In particular, the following variant of Gordan's theorem (replacing real numbers with integers) states that

$$\text{for any } M \in \mathbb{Z}^{m\times n},\ \textit{either}\ \ y^T M > 0 \ \text{ for some } y \in \mathbb{Z}^m \ \ \textit{or}\ \ Mx = 0 \ \text{ for some } x \in \mathbb{N}^n \backslash \{0\}.$$

This theorem is established in [4] by considering partial orders on free abelian groups and reformulated as the following correspondence between validity in the variety $\mathcal{LA}$ of abelian ℓ-groups of inequations $0 \leq t_1 \vee \ldots \vee t_n$, where $t_1, \ldots, t_n$ are group terms, and equations in the variety $\mathcal{A}$ of abelian groups:

$$\mathcal{LA} \models 0 \leq t_1 \vee \ldots \vee t_n \iff \mathcal{A} \models 0 \approx \lambda_1 t_1 + \cdots + \lambda_n t_n \ \text{ for some } \lambda_1, \ldots, \lambda_n \in \mathbb{N} \text{ not all } 0.$$

[1] Note that a very different approach to tackling this problem, via representations of totally ordered involutive commutative residuated lattices using ordered groups, has been described recently in [7].

This result may also be understood as generating a one-sided variant of the hypersequent calculus for abelian ℓ-groups introduced in [10].

In this work, we consider how far such theorems of alternatives can be extended to other classes of algebras and corresponding non-classical logics. Let $\mathcal{L}_g$ be the language of abelian groups with connectives $+$, $-$, 0, defining also $a \to b := -a + b$, $a \cdot b := -(-a + -b)$, $1 := -0$, and, inductively, for $n \in \mathbb{N}$, $0a := 0$, $a^0 := 1$, $(n+1)a = na + a$, and $a^{n+1} = a^n \cdot a$. We take as our starting point the following axiomatization of *Multiplicative Linear Logic* (MLL):

(B)	$(s \to t) \to ((t \to u) \to (s \to u))$	(1 L)	$s \to (1 \to s)$
(C)	$(s \to (t \to u)) \to (t \to (s \to u))$	(1 R)	1
(I)	$s \to s$	($-$L)	$-s \to (s \to 0)$
(INV)	$((s \to 0) \to 0) \to s$	($-$R)	$(s \to 0) \to -s$
($\cdot$ L)	$(s \to (t \to u)) \to ((s \cdot t) \to u)$	(+L)	$(s + t) \to -(-s \cdot -t)$
($\cdot$ R)	$s \to (t \to (s \cdot t))$	(+R)	$-(-s \cdot -t) \to (s + t)$

$$\frac{s \quad s \to t}{t}\ \text{(mp)}$$

We also define $\mathrm{MLL}_{0=1}$ to be the extension of MLL_0 with the axioms $0 \to 1$ and $1 \to 0$.

Algebraic semantics for MLL and its extensions are provided by *involutive commutative residuated pomonoids*: algebras $\langle A, +, -, 0, \leq \rangle$ satisfying (i) $\langle A, +, 0\rangle$ is a commutative monoid, (ii) $-$ is an involution on $\langle A, \leq \rangle$, (iii) $\leq$ is a partial order on $\langle A, +, 0\rangle$, and (iv) $a \cdot b \leq c \iff a \leq b \to c$ for all $a, b, c \in A$. For any axiomatic extension L of MLL, let $\mathcal{V}_{\mathrm{L}}$ be the class of involutive commutative residuated pomonoids satisfying $1 \leq s$ whenever $\vdash_{\mathrm{L}} s$. Then for any set of $\mathcal{L}_g$-terms $\Sigma \cup \{s\}$,

$$\Sigma \vdash_{\mathrm{L}} s \iff \{1 \leq t \mid t \in \Sigma\} \models_{\mathcal{V}_{\mathrm{L}}} 1 \leq s.$$

Let L^ℓ be the logic defined over the language $\mathcal{L}$ with connectives $+, -, 0, \wedge, \vee$ obtained by extending the axiomatization of L with the following axiom schema and rule:

($\wedge$1)	$(s \wedge t) \to s$	($\vee$1)	$s \to (s \vee t)$
($\wedge$2)	$(s \wedge t) \to t$	($\vee$2)	$t \to (s \vee t)$
($\wedge$3)	$((s \to t) \wedge (s \to u)) \to (s \to (t \wedge u))$	($\vee$3)	$((s \to u) \wedge (t \to u)) \to ((s \vee t) \to u)$
(PRL)	$(s \to t) \vee (t \to s)$	(DIS)	$((s \wedge (t \vee u)) \to ((s \wedge t) \vee (s \wedge u))$

$$\frac{s \quad t}{s \wedge t}\ \text{(adj)}$$

In particular, MLL^ℓ is involutive uninorm logic IUL formulated without the constants $\perp$ and $\top$ (see [9]).

Let $\mathcal{V}^\ell_{\mathrm{L}}$ be the variety generated by the totally ordered members of $\mathcal{V}_{\mathrm{L}}$ equipped with the binary meet and join operations $\wedge$ and $\vee$. Then for any set of $\mathcal{L}$-terms $\Sigma \cup \{s\}$,

$$\Sigma \vdash_{\mathrm{L}^\ell} s \iff \{1 \leq t \mid t \in \Sigma\} \models_{\mathcal{V}^\ell_{\mathrm{L}}} 1 \leq s.$$

Note that if $\mathcal{V}_{\mathrm{L}}$ is axiomatized over the class of involutive commutative residuated pomonoids by a set of equations E, then $\mathcal{V}^\ell_{\mathrm{L}}$ is axiomatized by E over the variety of involutive commutative semilinear residuated lattices.

We say that an axiomatic extension L of MLL admits a *theorem of alternatives* if for any set of $\mathcal{L}_g$-terms $\Sigma \cup \{t_1, \dots, t_n\}$,

$$\Sigma \vdash_{\mathrm{L}^\ell} t_1 \vee \ldots \vee t_n \iff \Sigma \vdash_{\mathrm{L}} \lambda_1 t_1 + \cdots + \lambda_n t_n \ \text{for some } \lambda_1, \dots, \lambda_n \in \mathbb{N} \text{ not all } 0.$$

This property can also be reformulated as a conservative extension property for L^ℓ over L.

Proposition 2.1. *An axiomatic extension* L *of* MLL *admits a theorem of alternatives if and only if* $\vdash_{\mathrm{L}^\ell} x \vee -x$ *and for any set of* $\mathcal{L}_g$*-terms* $\Sigma \cup \{t\}$,

$$\Sigma \vdash_{\mathrm{L}^\ell} t \iff \Sigma \vdash_{\mathrm{L}} \lambda t \text{ for some } \lambda \in \mathbb{N}\backslash\{0\}.$$

Note that the condition $\vdash_{\mathrm{L}^\ell} x \vee -x$ is immediate when L is an axiomatic extension of $\mathrm{MLL}_{0=1}$, and we will therefore assume this in what follows (even when more general results can be formulated).

The next result provides characterizations of logics admitting theorems of alternatives in terms of both consequences and valid formulas.

Theorem 2.2. *An axiomatic extension* L *of* $\mathrm{MLL}_{0=1}$ *admits a theorem of alternatives if and only if for all* $n \in \mathbb{N}\backslash\{0\}$,

$$\{nx,\, n(-x)\} \vdash_{\mathrm{L}} x^n + (-x)^n,$$

or, equivalently, if for all $n \in \mathbb{N}$*, there exist* $m \in \mathbb{N}\backslash\{0\}$*,* $k \in \mathbb{N}$ *such that* $\vdash_{\mathrm{L}} (nx)^k \to mx^n$.

In particular, any axiomatic extension L of the logic obtained by extending $\mathrm{MLL}_{0=1}$ with the axiom schema $nx \to x^n$ $(n \in \mathbb{N}\backslash\{0\})$ admits a theorem of alternatives. Moreover, the corresponding varieties $\mathcal{V}_{\mathrm{L}}^\ell$ of semilinear residuated lattices are exactly those axiomatized by group equations over the variety of involutive commutative semilinear residuated lattices satisfying $0 \approx 1$ and $nx \approx x^n$ $(n \in \mathbb{N}\backslash\{0\})$. These include the varieties of abelian ℓ-groups and odd Sugihara monoids.

3 Densifiability

We make use of the following lemma, originating in [9] (see also [2, 1, 11, 6, 3]).

Lemma 3.1. *A variety* $\mathcal{V}$ *of commutative semilinear residuated lattices is densifiable if and only if for any* $\mathcal{L}_g$*-terms* $s, t, u_1, \ldots, u_n$ *not containing the variable* x,

$$\mathcal{V} \models 1 \leq (s \to x) \vee (x \to t) \vee u_1 \vee \ldots \vee u_n \implies \mathcal{V} \models 1 \leq (s \to t) \vee u_1 \vee \ldots \vee u_n.$$

Consider any axiomatic extension L of $\mathrm{MLL}_{0=1}$ that admits a theorem of alternatives. Suppose that $\mathcal{V}_{\mathrm{L}}^\ell \models 1 \leq (s \to x) \vee (x \to t) \vee u_1 \vee \ldots \vee u_n$ where $s, t, u_1, \ldots, u_n$ are $\mathcal{L}_g$-terms not containing the variable x. Since L admits a theorem of alternatives, there exist $\lambda, \mu, \gamma_1, \ldots, \gamma_n \in \mathbb{N}$ not all 0 such that

$$\mathcal{V}_{\mathrm{L}}^\ell \models 1 \leq \lambda(s \to x) + \mu(x \to t) + \gamma_1 u_1 + \ldots + \gamma_n u_n.$$

Substituting, on the one hand x with 0, and on the other, all other variables with 0, yields

$$\mathcal{V}_{\mathrm{L}}^\ell \models 1 \leq \lambda(-s) + \mu t + \gamma_1 u_1 + \ldots + \gamma_n u_n \quad \text{and} \quad \mathcal{V}_{\mathrm{L}}^\ell \models 1 \leq \lambda x + \mu(-x).$$

Substituting x with s^λ in the second inequation and rewriting both inequations then yields

$$\mathcal{V}_{\mathrm{L}}^\ell \models \lambda(s^\lambda) \leq \lambda\mu t + \lambda\gamma_1 u_1 + \ldots + \lambda\gamma_n u_n \quad \text{and} \quad \mathcal{V}_{\mathrm{L}}^\ell \models s^{\lambda\mu} \leq \lambda(s^\lambda).$$

By transitivity, we obtan $\mathcal{V}_{\mathrm{L}}^\ell \models s^{\lambda\mu} \leq \lambda\mu t + \lambda\gamma_1 u_1 + \ldots + \lambda\gamma_n u_n$, which can be rewritten as

$$\mathcal{V}_{\mathrm{L}}^\ell \models 1 \leq \lambda\mu(s \to t) + \lambda\gamma_1 u_1 + \ldots + \lambda\gamma_n u_n.$$

But then, by the theorem of alternatives,

$$\mathcal{V}_{\mathrm{L}}^\ell \models 1 \leq (s \to t) \vee u_1 \vee \ldots \vee u_n.$$

Hence, by the lemma, we obtain our main result.

Theorem 3.2. *Let* L *be any axiomatic extension of* $\mathrm{MLL}_{0=1}$ *that admits a theorem of alternatives. Then the variety* $\mathcal{V}_{\mathrm{L}}^{\ell}$ *is densifiable.*

In particular, any variety $\mathcal{V}_{\mathrm{L}}^{\ell}$ axiomatized by group equations over the variety of involutive commutative semilinear residuated lattices satisfying $0 \approx 1$ and $nx \approx x^n$ ($n \in \mathbb{N}\backslash\{0\}$) is densifiable, including (as is already well-known) the varieties of abelian ℓ-groups and odd Sugihara monoids.

References

[1] P. Baldi and K. Terui. Densification of FL-chains via residuated frames. *Algebra Universalis*, 75:169–195, 2016.

[2] A. Ciabattoni and G. Metcalfe. Density elimination. *Theoret. Comput. Sci.*, 403:328–346, 2008.

[3] P. Cintula and C. Noguera. Implicational (semilinear) logics iii: completeness properties. *Arch. Math. Logic*, 57(3):391–420, 2018.

[4] A. Colacito and G. Metcalfe. Proof theory and ordered groups. In *Proc. WoLLIC 2017*, volume 10388 of *LNCS*, pages 80–91. Springer, 2017.

[5] G. B. Dantzig. *Linear programming and extensions*. Princeton Univ. Press, 1963.

[6] N. Galatos and R. Horčik. Densification via polynomials, languages, and frames. *J. Pure Appl. Algebra*, to appear.

[7] S. Jenei. Group representation, Hahn-type embedding, and the FSSC for IUL^{fp}. In *Proc. LATD 2018*, pages 63–66, 2018.

[8] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's MTL logic. *Studia Logica*, 70(2):183–192, 2002.

[9] G. Metcalfe and F. Montagna. Substructural fuzzy logics. *J. Symbolic Logic*, 72(3):834–864, 2007.

[10] G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Trans. Comput. Log.*, 6(3):578–613, 2005.

[11] G. Metcalfe and C. Tsinakis. Density revisited. *Soft Comput.*, 21:175–189, 2017.