Universal Objects for Orders on Groups, and their Dual Spaces

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Following on from the success of scheme theory in algebraic geometry, Keimel in 1971 introduced in his doctoral dissertation [10] a notion of spectral space associated to Abelian lattice-ordered groups (cf. [2, Chapter 10]). For an Abelian lattice-ordered group H, the ℓ-spectrum is defined as the set of its prime ℓ-ideals with the spectral topology. The notion of ℓ-spectrum is not limited to the commutative setting, and can also be defined for an arbitrary lattice-ordered group (see, e.g., [6] and [7, Chapter 9]).

A *lattice-ordered group* (briefly, *ℓ-group*) H is a group with a lattice structure compatible with the group operation, i.e., the group operation distributes over the lattice operations. We call an ℓ-group *representable* if it is a subdirect product of chains, and *Abelian* if its underlying group is Abelian. The *ℓ-spectrum* $\operatorname{Spec} H$ of an ℓ-group H is the root system of all its prime convex ℓ-subgroups ordered by inclusion. Here, a *convex ℓ-subgroup* of H is an order-convex sublattice subgroup of H, while an *ℓ-ideal* is a convex ℓ-subgroup which is also normal, i.e., closed under conjugation. For any convex ℓ-subgroup K, the quotient H/K is lattice-ordered by: $\mathrm{K}x \leqslant_{H/\mathrm{K}} \mathrm{K}y$ if, and only if, there exists $t \in \mathrm{K}$ such that $x \leqslant ty$. A convex ℓ-subgroup P is *prime* when the quotient H/P is non-trivial and totally ordered. A prime convex ℓ-subgroup of H is *minimal* if it is inclusion-minimal in $\operatorname{Spec} H$. By an application of Zorn's Lemma, any prime convex ℓ-subgroup of H contains a minimal prime convex ℓ-subgroup. We write $\operatorname{Min} H$ for the set of minimal prime convex ℓ-subgroups of H. Given an ℓ-group H, we consider the following *hull-kernel* (or *spectral*, or *Stone*, or *Zariski*) topology on $\operatorname{Spec} H$. The basic open sets are

$$\mathbb{S}_x = \{\mathrm{P} \in \operatorname{Spec} H \mid x \notin \mathrm{P}\}, \quad \text{for } x \in H$$

and we refer to $\mathbb{S}_x$ as the *support* of $x \in H$. We also endow $\operatorname{Min} H$ with the subspace topology. It can be proved that $\operatorname{Min} H$ is Hausdorff [7, Proposition 49.8].

We adopt the standard notation $x \perp y$—read 'x and y are orthogonal'—to denote $|x| \wedge |y| = e$ for $x, y \in H$, where $|x| = x \vee x^{-1}$ is the *absolute value* of x, and 'e' is the group identity. For $T \subseteq H$, set

$$T^{\perp} = \{x \in H \mid x \perp y \text{ for all } y \in T\},$$

and call those subsets *polars*. We write $\operatorname{Pol} H$ for the Boolean algebra of polars of H, and $\operatorname{Pol}_{\mathrm{p}} H$ for its sublattice of principal polars, namely those of the form $\{x\}^{\perp\perp}$ for some $x \in H$.

The spectral space of an ℓ-group H provides—in the case in which H is representable—a tool for employing sheaf-theoretic methods in the study of ℓ-groups.

Example. A representable ℓ-group H can be embedded into a Hausdorff sheaf of ℓ-groups on the Stone space associated with the complete Boolean algebra $\operatorname{Pol} H$ of polars [7, Proposition 49.21].

Furthermore, topological properties of the ℓ-spectrum $\operatorname{Spec} H$ can have important consequences on the structure of the ℓ-group H.

Example. The space $\operatorname{Min} H$ is compact if, and only if, $\operatorname{Pol}_{\mathrm{p}} H$ is a Boolean algebra [6].[1]

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[1] Further striking examples—although not relevant for the present abstract—are the following: $\operatorname{Spec} H$ is Hausdorff if, and only if, H is hyperarchimedean [6, 1.2]; $\operatorname{Spec} H$ is compact if, and only if, H has a strong order unit [6, 1.3].

In 2004, Sikora's paper 'Topology on the spaces of orderings of groups' [13] pioneered a different perspective on the study of the interplay between topology and ordered groups, that has led to applications to both orderable groups and algebraic topology (see, e.g., [3]). The basic construction in Sikora's paper is the definition of a topology on the set of right orders on a given right orderable group.

A binary relation R on a group G is *right-invariant* (resp. *left-invariant*) if for all $a, b, t \in G$, whenever aRb then $atRbt$ (resp. $taRtb$). We call a binary relation $\leqslant$ on a set S a (*total*) *order* if it is reflexive, transitive, antisymmetric, and total. A *right order on* G is just a right-invariant order on G, and G is *right orderable* if there exists a right order on G. A submonoid $C \subseteq G$ is a (*total*) *right cone for* G if $G = C \cup C^{-1}$ and $\{e\} = C \cap C^{-1}$. We set $\mathcal{R}(G)$ to be the set of right cones for G. It is elementary that $\mathcal{R}(G)$ is in bijection with the right orders on G via the map that associates to $C \in \mathcal{R}(G)$ the relation: $a \leqslant_C b$ if, and only if, $ba^{-1} \in C$. Hence, we refer to $\mathcal{R}(G)$ as 'the set of right orders on G'.

For a right orderable group G, Sikora endowed $\mathcal{R}(G)$ with the subspace topology inherited from the power set 2^G with the Tychonoff topology. A subbasis of clopens for $\mathcal{R}(G)$ is given by the sets

$$\mathbb{R}_a = \{C \in \mathcal{R}(G) \mid a \in C\}, \quad \text{for } a \in G.$$

The subspace $\mathcal{R}(G)$ can be proved to be closed in 2^G, and is therefore a compact totally disconnected Hausdorff space, i.e., it is a Stone space.

The theory of ℓ-groups and the theory of right orderable groups have been proved to be deeply related, and examples of this interdependence can be found almost everywhere in the literature of either field (see, e.g., [9, 11, 8, 4]). For this reason, the question whether a relation can be found between the topological space of right orders on a right orderable group G, and the ℓ-spectrum of some ℓ-group H arises naturally. In this work, we provide a positive answer to this question. In order to give a satisfying result that intrinsically relates the two topological spaces, we employ a fully general and natural construction, involving all the varieties of ℓ-groups. We focus here on the particular result, and only briefly sketch how the latter fits into the general framework.

For a group G, we write $F(G)$ for the free ℓ-group over G (as a group), and $\eta_G \colon G \to F(G)$ for the group homomorphism characterized by the following universal property: for each group homomorphism $p \colon G \to H$, with H an ℓ-group, there is exactly one ℓ-homomorphism $h \colon F(G) \to H$ such that $h \circ \eta_G = p$, i.e., such that the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & F(G) \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle !h} \\ & & H \end{array}$$

commutes. As it turns out, $F(G)$ is the ℓ-group that we were looking for.

Theorem 1. *Given a right orderable group G, the space $\mathcal{R}(G)$ of right orders on G is homeomorphic to the minimal layer $\operatorname{Min} F(G)$ of the ℓ-spectrum $\operatorname{Spec} F(G)$. As a consequence, the lattice $\operatorname{Pol_p} F(G)$ of principal polars is a Boolean algebra and $\mathcal{R}(G)$ is its dual Stone space.*

We obtain Theorem 1 as a consequence of a result involving the whole of $\operatorname{Spec} F(G)$. For this, it is necessary to consider the broader notion of right pre-order. A binary relation $\preccurlyeq$ on a set S is a (*total*) *pre-order* if it is reflexive, transitive, and total. A *right pre-order on* G is a right-invariant pre-order on G. A submonoid $C \subset G$ is a (*total*) *right pre-cone for* G if $G = C \cup C^{-1}$. We set $\mathcal{P}(G)$ to be the root system of total right pre-cones for G ordered by inclusion. It is again elementary that $\mathcal{P}(G)$ is in bijection with the right pre-orders on G via the map that associates to $C \in \mathcal{P}(G)$ the relation: $a \preccurlyeq_C b$ if, and only if, $ba^{-1} \in C$. Note that if the group G is right orderable, then Sikora's $\mathcal{R}(G)$ is a subset of $\mathcal{P}(G)$. More precisely, if $\mathcal{R}(G) \neq \emptyset$, it can be proved to coincide with the minimal layer of $\mathcal{P}(G)$.

For any group G, we set

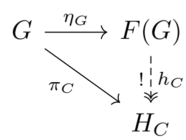
$$\mathbb{P}_a = \{C \in \mathcal{P}(G) \mid a \in C \text{ and } a \notin C^{-1}\}, \quad \text{for } a \in G$$

and endow $\mathcal{P}(G)$ with the smallest topology for which all sets $\mathbb{P}_a$ are open. Thus, any open set in this topology is the union of sets of the form $\mathbb{P}_{a_1} \cap \ldots \cap \mathbb{P}_{a_n}$. If G is right orderable, the subspace topology on $\mathcal{R}(G)$ amounts to Sikora's topology, and hence, the minimal layer of $\mathcal{P}(G)$ is compact.

Observe that there is a natural way to obtain an ℓ-group from a right pre-cone C for G. In fact, the set $G_{\gtreqless} = C \cap C^{-1}$ is a subgroup of G, and the quotient $G/G_{\gtreqless}$ can be totally ordered by: $[a] \leqslant [b]$ if, and only if, $a \preccurlyeq_C b$; if Ω_C is the resulting chain, we can consider the ℓ-subgroup H_C of $Aut(\Omega_C)$ generated by the image of G through the group homomorphism $\pi_C \colon G \to Aut(\Omega_C)$ defined by

$$a \mapsto (\pi_C(a)[b] = [ba]).$$

Note that the ℓ-group H_C allows us to exploit the universal property of $\eta_G \colon G \to F(G)$, inducing the existence of the surjective ℓ-group homomorphism h_C, making the following diagram



commute. We then make use of h_C to conclude the following result.

Proposition 1. *The topological spaces $\mathcal{P}(G)$ and $\operatorname{Spec} F(G)$ are homeomorphic.*

We would like to point out that a correspondence between the two root systems from Proposition 1 for the particular case in which G is a free group can essentially be found in [12].

The proof of Proposition 1 is self-contained, and only uses basic facts about (lattice-)ordered groups. An order-isomorphism between the underlying root systems is built explicitly, and then proved to be a homeomorphism between the corresponding topological spaces. Besides Theorem 1, a further consequence of Proposition 1 is an alternative proof of a fundamental representation theorem for free ℓ-groups, originally proved by Conrad [5]. As already remarked, Proposition 1 is an instance of a much more general result associating a family of right pre-orders to each variety V of ℓ-groups: given a group G, the right pre-orders on G associated with the variety V are exactly those for which $H_C \in \mathsf{V}$.

Without going into details, we conclude by stating a further consequence of the above-mentioned general result. Given a group G, we write $F(G)_{\mathsf{R}}$ for the free representable ℓ-group over the group G and $\eta_G \colon G \to F(G)_{\mathsf{R}}$ for the corresponding universal morphism (i.e., characterized by the universal property with respect to the variety R of representable ℓ-groups). An *order on* G is just a left-invariant right order on G, and G is *orderable* if there exists an order on G. As in the case of right orders, the set $\mathcal{O}(G)$ of orders on a group G can be identified with a set of subsets of G, namely those right cones that are closed under conjugation. Sikora's topological space on $\mathcal{O}(G)$ can then be defined as the smallest topology containing the sets $\{C \in \mathcal{O}(G) \mid a \in C\}$, for $a \in G$.

Theorem 2. *Given an orderable group G, the space $\mathcal{O}(G)$ of orders on G is homeomorphic to the minimal layer $\operatorname{Min} F(G)_{\mathsf{R}}$ of the ℓ-spectrum $\operatorname{Spec} F(G)_{\mathsf{R}}$. As a consequence, the lattice $\operatorname{Pol_p} F(G)_{\mathsf{R}}$ of principal polars is a Boolean algebra and $\mathcal{O}(G)$ is its dual Stone space.*

Theorem 2 is, similarly to Theorem 1, a consequence of the existence of a homeomorphism between the space of representable right pre-orders on G, namely those for which $H_C \in \mathsf{R}$, and the ℓ-spectrum $\operatorname{Spec} F(G)_{\mathsf{R}}$. Its main significance lies in the fact that it provides a new perspective on some open problems in the theory of orderable groups (e.g., it is still unknown whether the equational theory of R is decidable; it is also unknown whether the space $\mathcal{O}(G)$, for G free of finite rank $n \geq 2$, is Cantor [1]).

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