An Ordering Condition for Groups

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Ordering conditions for groups provide useful tools for the study of various relationships between group theory, universal algebra, and topology (see, e.g., [2, 4, 3, 1]). In this work, we establish a new "algorithmic" ordering condition for extending partial orders on groups to total orders. We then use this condition to show that the problem of extending a finite subset of a free group to a total order corresponds to the problem of checking validity of a certain inequation in the variety of representable lattice-ordered groups (or, equivalently, the class of totally ordered groups). As a direct consequence, we obtain a new proof that free groups are orderable.

Let us fix a group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$. Recall that a *partial order* of $\mathbf{G}$ is a partial order $\leq$ on G satisfying also for $a, b, c, d \in G$,

$$a \leq b \implies cad \leq cbd.$$

Its positive cone $P_{\leq} = \{a \in G : e < a\}$ is a normal subsemigroup of $\mathbf{G}$ (a subset of G closed under $\cdot$ and conjugation by elements of G) that omits e. Conversely, if P is a normal subsemigroup of $\mathbf{G}$ omitting e, then $\mathbf{G}$ is partially ordered by

$$a \leq^P b \iff ba^{-1} \in P \cup \{e\}.$$

Hence partial orders of $\mathbf{G}$ can be identified with normal subsemigroups of $\mathbf{G}$ not containing e. For $S \subseteq G$, the normal subsemigroup of $\mathbf{G}$ generated by S, denoted by $\langle\langle S \rangle\rangle$, is a partial order of $\mathbf{G}$ if and only if $e \notin \langle\langle S \rangle\rangle$. A partial order $\leq$ of $\mathbf{G}$ is a *(total) order* if $G = P_{\leq} \cup {P_{\leq}}^{-1} \cup \{e\}$.

Now, for finite subsets $S \subseteq G$, we define a relation $\vdash_{\mathbf{G}} S$ inductively by the clauses

(i) $\vdash_{\mathbf{G}} S \cup \{a, a^{-1}\}$;

(ii) $\vdash_{\mathbf{G}} S \cup \{ab\}$, whenever $\vdash_{\mathbf{G}} S \cup \{a\}$ and $\vdash_{\mathbf{G}} S \cup \{b\}$;

(iii) $\vdash_{\mathbf{G}} S \cup \{ab\}$, whenever $\vdash_{\mathbf{G}} S \cup \{ba\}$.

The following theorem describes our new condition for extending a finite subset of $\mathbf{G}$ to an order, noting that the equivalence of (1) and (2) is a reformulation of an ordering theorem for groups due to Fuchs [2].

Theorem 1. *The following are equivalent for a finite $S \subseteq G$:*

(1) *S does not extend to a total order of $\mathbf{G}$.*

(2) *There exist $a_1, \ldots, a_m \in G \setminus \{e\}$ such that for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$,*

$$e \in \langle\langle S \cup \{a_1^{\delta_1}, \ldots, a_m^{\delta_m}\} \rangle\rangle.$$

(3) $\vdash_{\mathbf{G}} S$.

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We now consider a non-trivial free group $\mathbf{F}$, which may be viewed as an algebra of reduced group terms obtained by cancelling all the occurrences of xx^{-1} and $x^{-1}x$. For convenience, we deliberately confuse group terms t with their counterparts in $\mathbf{F}$. We consider also the variety $\mathcal{RG}$ of representable lattice-ordered groups (in an algebraic language with operations $\wedge, \vee, \cdot, ^{-1}, e$) generated by the class of totally ordered groups. Using Theorem 1, we then obtain the following correspondence between extending a finite subset of $\mathbf{F}$ to an order and the validity of a corresponding inequation in $\mathcal{RG}$.

Theorem 2. *The following are equivalent for any $t_1, \ldots, t_n \in F$:*

(1) $\{t_1, \ldots, t_n\}$ *does not extend to a total order of* $\mathbf{F}$.

(2) $\vdash_{\mathbf{F}} \{t_1, \ldots, t_n\}$.

(3) $\mathcal{RG} \models e \leq t_1 \vee \ldots \vee t_n$.

This result is then used to obtain a new proof of the orderability of free groups, first proved in [5]. In fact, it is sufficient to observe that $\mathcal{RG} \not\models e \leq x$ for any generator x, and hence, by Theorem 2, there exists an order of $\mathbf{F}$ where x is positive.

References

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