Orders on Groups: an Approach through Spectral Spaces

Almudena Colacito*

Mathematisches Institut, Universität Bern
almudena.colacito@math.unibe.ch

A *right order* on a group G is a total order $\leq$ on G such that $x \leq y$ implies $xt \leq yt$, for all $x, y, t \in G$. There are at least two distinct ways that could lead a mathematician to encounter such orders on groups. First, it is standard that a countable group admits a right order if, and only if, it acts faithfully on the real line by orientation-preserving homeomorphisms (see, e.g., [7, Theorem 6.8]). The result indicates that orders on groups play a rôle in topological dynamics. Second, right orders are central to the theory of *lattice-ordered groups* (briefly, *ℓ-groups*), i.e., groups with a lattice structure compatible with the group operation (see, e.g., [9, 8]). For a recent connection between right orders and ℓ-group equations, see [4].

In 2004, Sikora's paper "Topology on the spaces of orderings of groups" [11] pioneered a different perspective on the study of the interplay between topology and ordered groups, that has led to applications to both orderable groups and algebraic topology. The basic construction in Sikora's paper is the definition of a topology on the set of right orders $\mathcal{R}(G)$ on a group G, thereby associating a natural topological space to any right-orderable group. The space is then proved compact, Hausdorff, and zero-dimensional (see, e.g., [2, Problem 1.38]).

We show in [3] that Sikora's space arises naturally from the study of ℓ-groups, as the minimal spectrum of the ℓ-group freely generated by the group at hand. The *ℓ-spectrum* $\operatorname{Spec} H$ of an ℓ-group H is the root system of all its prime subgroups ordered by inclusion topologised with the hull-kernel topology. Here, a *prime subgroup* of H is an order-convex sublattice subgroup $\mathfrak{p}$ of H with the further property that $x \wedge y \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. We write $\operatorname{Min} H$ for the set of inclusion-minimal prime subgroups of H with the subspace topology. By an application of Zorn's Lemma, any prime subgroup of H contains a minimal prime subgroup. It can be proved that $\operatorname{Min} H$ is Hausdorff [6, Proposition 49.8], and that compactness of $\operatorname{Spec} H$ is equivalent to the existence of a *strong order unit*—an element $u \in H^+ := \{x \in H \mid \mathrm{e} \leq x\}$ that generates H as an order-convex sublattice subgroup, where e denotes the group identity [5, 1.3]. If H is finitely generated, every $\mathfrak{p} \in \operatorname{Spec} H$ is included in a unique maximal prime subgroup, and we write $\operatorname{Max} H$ for the set of maximal prime subgroups of H with the subspace topology. We call an ℓ-group *representable* if it is a subdirect product of totally ordered groups.

By replacing right orders with right *pre*-orders—pre-orders that are invariant under group multiplication on the right—we provide a systematic, structural account of the relationship between (total) right pre-orders on a group G and prime subgroups of the ℓ-group $F^\ell(G)$ freely generated by G (or over G). This connection is developed in full generality—that is, for any variety of ℓ-groups (see [3, Theorem 2.1]). It follows that the space of right pre-orders on any group G is homeomorphic to the ℓ-spectrum $\operatorname{Spec} F^\ell(G)$ and, when G is right orderable, Sikora's space $\mathcal{R}(G)$ is homeomorphic to $\operatorname{Min} F^\ell(G)$. Further, when G admits an *order* (i.e., a right order that is invariant under group multiplication on the left), the minimal spectrum of the free representable ℓ-group $F^\ell_{\mathsf{R}}(G)$ over G is homeomorphic to the space $\mathcal{O}(G)$ of orders on G. This theoretical framework leads to a few immediate, diverse results, some of which are listed below.

Example. The space of right pre-orders on a finitely generated group G is compact.

This follows immediately from the fact that the finitely many generators of G induce the existence of a strong order unit on $F^\ell(G)$.

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Example. For any group G, the space $\operatorname{Min} F^\ell(G)$ is compact.

In fact, we show that the space of right pre-orders on G is either empty or homeomorphic to the space of right pre-orders on a corresponding right-orderable group G^* [3, Remark 6.2]. The minimal layer of the latter is Sikora's $\mathcal{R}(G^*)$, which is indeed compact. This topological property of the minimal spectrum can be reformulated algebraically as follows. For any ℓ-group H, we say that H^+ is *complemented* if for every $x \in H^+$ there is a $y \in H^+$ such that $x \wedge y = \mathrm{e}$, and $x \vee y$ is a *weak order unit*—an element $w \in H^+$ such that $w \wedge x = \mathrm{e}$ implies $x = \mathrm{e}$.

Example. For any group G, the distributive lattice $F^\ell(G)^+$ is complemented.

Example. The free ℓ-group $F^\ell(n)$ of rank $n \geq 2$ acts by homeomorphism on the Cantor space.

For this, it suffices to observe that $F^\ell(n)$ acts by homeomorphism on $\operatorname{Min} F^\ell(n)$. It follows from [10, Corollary 5], [2, §1.5.2], and [3], that $\operatorname{Min} F^\ell(n)$ is the Cantor space.

These consequences are nothing more than translations of results from the right order setting to the ℓ-group setting, or vice versa, facilitated by the correspondence established in [3]. We mention here two possibilities for further development.

First, we observe that [3] provides a new perspective on the open question whether there exist isolated points in $\mathcal{O}(F(n))$ for $n \geq 2$ (this question was raised by McCleary in [1, §4]). More precisely, we get a necessary condition for the existence of such isolated points. In fact, it is possible to argue that $\operatorname{Max} F^\ell_{\mathsf{R}}(n)$ is the $(n-1)$-sphere $\mathbb{S}^{n-1}$, and that there exists a closed continuous map $\lambda\colon \operatorname{Min} F^\ell_{\mathsf{R}}(n) \twoheadrightarrow \operatorname{Max} F^\ell_{\mathsf{R}}(n)$. Thus, if λ is irreducible—it sends proper closed subsets to proper closed subsets—then $\operatorname{Min} F^\ell_{\mathsf{R}}(n)$ has no isolated points. We finally mention the intriguing problem of obtaining a natural representation of $F^\ell_{\mathsf{R}}(n)$ for $n \geq 2$ in terms of (possibly, piecewise linear) functions. In tackling this problem, we believe that the connection between topological dynamics and right orders mentioned in the beginning will be a key ingredient.

References

[1] Ashok K. Arora and Stephen H. McCleary. Centralizers in free lattice-ordered groups. *Houston J. Math., 12*, 1986.

[2] Adam Clay and Dale Rolfsen. *Ordered Groups and Topology*, volume 176 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016.

[3] Almudena Colacito and Vincenzo Marra. Orders on groups, and spectral spaces of lattice-groups. arXiv preprint arXiv:1901.07638. 2019.

[4] Almudena Colacito and George Metcalfe. Ordering groups and validity in lattice-ordered groups. *J. Pure Appl. Algebra*. To appear. 2019.

[5] Paul Conrad and Jorge Martinez. Complemented lattice-ordered groups. *Indag. Math.*, 1(3):281–297, 1990.

[6] Michael R. Darnel. *Theory of Lattice-Ordered Groups*, volume 187 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1995.

[7] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.

[8] Andrew M. W. Glass. *Partially Ordered Groups*, volume 7. World Scientific, 1999.

[9] Valerii M. Kopytov and Nikolai Ya. Medvedev. *The Theory of Lattice-Ordered Groups*, volume 307 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1994.

[10] Stephen H. McCleary. Free lattice-ordered groups represented as *o*-2 transitive ℓ-permutation groups. *Trans. Am. Math. Soc.*, 290(1):69–79, 1985.

[11] Adam S. Sikora. Topology on the spaces of orderings of groups. *Bull. Lond. Math. Soc.*, 36(4):519–526, 2004.