The Dimension of an Amoeba

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ASGARD Math 2019

May 16, 2019

Basic Setup

- $X \subset (\mathbb{C}^*)^n$: irreducible subvariety.
- $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ by $(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$.
- *Ameoba* $\mathcal{A}(X) := \mathrm{Log}(X)$.
- Notion by Gelfand–Kapranov–Zelevinsky. Related to A-discriminants, real algebraic geometry, mirror symmetry, etc.

Example of Amoeba

$X = V(x + y + 1) \subset (\mathbb{C}^*)^2$.

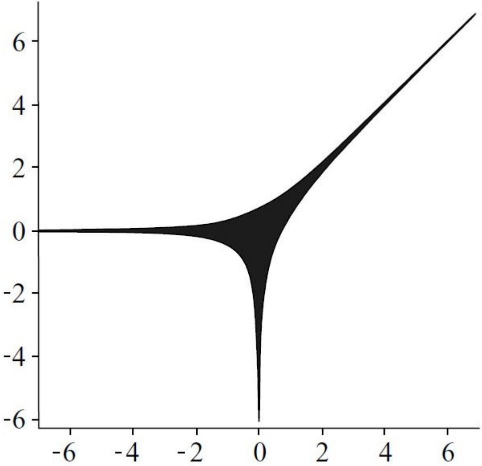
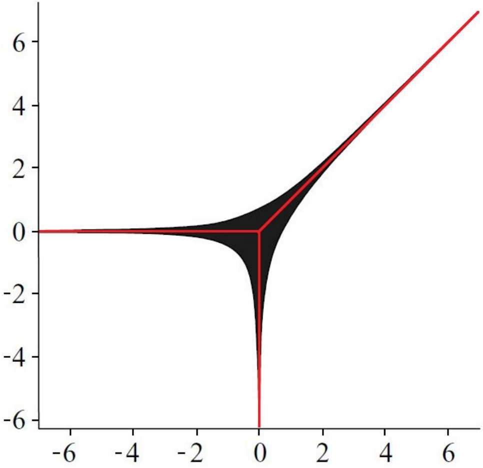


Figure 1.2 of *Tropical Algebraic Geometry* by Itenberg et al.

Tropical Connection

An amoeba has a canonical spine, which is the tropicalisation of X.



$\mathcal{A}(X)$ and $\text{Trop}(X)$.

Question

Well-known: $\dim_{\mathbb{R}} \mathrm{Trop}(X) = \dim_{\mathbb{C}} X$.

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What about the (real) dimension of $\mathcal{A}(X)$?

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- Intuition/generic case: $\dim_{\mathbb{R}} X = 2\dim_{\mathbb{C}} X$, and Log is "nice", so $\dim_{\mathbb{R}} \mathcal{A}(X) = 2\dim_{\mathbb{C}} X$.

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- Intuition/generic case: $\dim_{\mathbb{R}} X = 2\dim_{\mathbb{C}} X$, and Log is "nice", so $\dim_{\mathbb{R}} \mathcal{A}(X) = 2\dim_{\mathbb{C}} X$.
- In general $2\dim_{\mathbb{C}} X$ is an upper bound, but equality does not always hold.

Examples of Dimension Drop

Example (Hypersurface)

If $n > 2$ and X is a hypersurface, then
$\dim_{\mathbb{R}} \mathcal{A}(X) \leq \dim_{\mathbb{R}} \mathbb{R}^n = n < 2(n-1) = 2 \dim_{\mathbb{C}} X$.

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Example (Torus)

$X = \{(z^1w^4, z^2w^5, z^3w^6) : z, w \in \mathbb{C}^*\}$ is a 2-dimensional subtorus.
$\mathcal{A}(X) = \text{span}\{(1,2,3),(4,5,6)\}$ is a 2-dimensional subspace.
In general, the amoeba of a k-dimensional subtorus is a k-dimensional subspace.

More Examples of Dimension Drop

Example (Torus Action)

Suppose $S \cdot X := \{(s_1 z_1, \ldots, s_n z_n) : \mathbf{s} \in S, \mathbf{z} \in X\} = X$ for some k-dim torus S.
$X \mapsto X/S =: Y \subset (\mathbb{C}^*)^n/S$ (resp. $\mathcal{A}(X) \mapsto \mathcal{A}(X)/\mathcal{A}(S) = \mathcal{A}(Y)$) has fibers isomorphic to S (resp. $\mathcal{A}(S)$).
So $\dim_{\mathbb{R}} \mathcal{A}(X) = k + \dim_{\mathbb{R}} \mathcal{A}(Y) \leq k + 2\dim_{\mathbb{C}} Y = k + 2(\dim_{\mathbb{C}} X - k) = 2\dim_{\mathbb{C}} X - k$.

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- Nisse–Sottile (2018) suggested a program to understand amoebas better, including a conjecture about the dimension of amoebas.

Main Theorem

Theorem (Draisma–Rau–Y. 2018+)

$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2\dim_{\mathbb{C}} X + 2\dim_{\mathbb{C}} T - \dim_{\mathbb{C}} S\}$, *taken over* $T \subset S \subset (\mathbb{C}^*)^n$ *subtori such that* $S \cdot \overline{T \cdot X} = \overline{T \cdot X}$.

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Corollary (Conjecture of Nisse–Sottile)

$\dim_{\mathbb{R}} \mathcal{A}(X) < \min\{2\dim_{\mathbb{C}} X, n\}$ *iff* X *admits a* near/diminishing torus action.

Verifications

Theorem (DRY 2018+)

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2\dim_{\mathbb{C}} X + 2\dim_{\mathbb{C}} T - \dim_{\mathbb{C}} S : S \cdot \overline{T \cdot X} = \overline{T \cdot X}\}.$$

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Take $T = S = \{\mathbf{1}\}$. Then $\dim_{\mathbb{R}} \mathcal{A}(X) \leq 2\dim_{\mathbb{C}} X$.

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Example (Hypersurface)

Take T to be any generic 1-dim subtorus such that $\overline{T \cdot X} = (\mathbb{C}^*)^n$. Then $\dim_{\mathbb{R}} \mathcal{A}(X) \leq 2\dim_{\mathbb{C}} X + 2\dim_{\mathbb{C}} T - \dim_{\mathbb{C}}(\mathbb{C}^*)^n = 2(n-1) + 2 - n = n$.

Sketch of Proof: The Easy Half

Theorem (DRY 2018+)

$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2\dim_{\mathbb{C}} X + 2\dim_{\mathbb{C}} T - \dim_{\mathbb{C}} S : S \cdot \overline{T \cdot X} = \overline{T \cdot X}\}.$

Proof of $\leq$:

$$\begin{aligned}
&\dim_{\mathbb{R}} \mathcal{A}(X) \\
&\leq \dim_{\mathbb{R}} \mathcal{A}(\overline{T \cdot X}) \\
&\leq 2\dim_{\mathbb{C}} \overline{T \cdot X} - \dim_{\mathbb{C}} S \\
&\leq 2(\dim_{\mathbb{C}} X + \dim_{\mathbb{C}} T) - \dim_{\mathbb{C}} S.
\end{aligned}$$

Sketch of Proof: Overview of the Harder Half

- $\text{Abs} : (\mathbb{C}^*)^n \to \mathbb{R}^n_{>0}$ by $(z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|)$.
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- Goal: Find a *rational* subspace U of positive dimension that is contained in (almost) all $T_q\mathcal{A}(X)$'s.
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  Rational: $U = \text{span}(U \cap \mathbb{Q}^n)$.
- Idea: $\overline{|X|}$ is stable under the action of R, the real subtorus whose tangent space is U. T, S will be inductively constructed using R (and its complexification).

Sketch of Proof: Swapping Quantifiers

Goal: Find a rational subspace U contained in (almost) all $T_q\mathcal{A}(X)$'s.

Lemma ("Swapping Quantifiers Principle")

"$\exists U, \forall q, U \leq T_q\mathcal{A}(X)$" is equivalent to "$\forall q, \exists U_q, U_q \leq T_q\mathcal{A}(X)$".

Proof: Suppose $|X| \approx \mathcal{A}(X)$ equals the union of (real-Zariski-closed) $\{q : U \leq T_q\mathcal{A}(X)\}$ over all rational U's. $|X|$ is irreducible and the union is countable, so one of such $\{q : U \leq T_q\mathcal{A}(X)\}$'s is the whole of $|X|$.

Sketch of Proof: Swapping Real and Imaginary Parts

Since $z = re^{i\theta}$, each $T_{\mathbf{z}}X$ decomposes into real and imaginary parts from $T_{\mathbf{1}}(\mathbb{C}^*)^n = \mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n = T_{\mathbf{1}}\mathbb{R}^n_{>0} \oplus T_{\mathbf{0}}(S^1)^n$.

Observation

Abs *takes the real part to* $T_{|\mathbf{z}|}|X|$ *and kills the imaginary part. But* $T_{\mathbf{z}}X$ *is a complex v.s., so its real part is precisely* i *times its imaginary part.*

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- U is essentially the tangent space of $\langle Z_q \rangle$.

More Tropical Connection

Corollary

$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2\dim_{\mathbb{R}}(S + \mathrm{Trop}(X)) - \dim_{\mathbb{R}} S : S \leq \mathbb{R}^n \text{ rational}\}.$

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- The above formula is computable (using real quantifier elimination) if the rationality condition is dropped.

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Can $\dim_{\mathbb{R}} \mathcal{A}(X)$ *be computed given* $\mathrm{Trop}(X)$*?*

- The above formula is computable (using real quantifier elimination) if the rationality condition is dropped.
- But can we drop it?

Thank you!

Application

Proposition (Nisse–Sottile)

$\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{C}} X$ *iff* X *is a single torus orbit* $S \cdot \mathbf{x}$.

Proof: $2 \dim_{\mathbb{C}} \overline{S \cdot X} - \dim_{\mathbb{C}} S = \dim_{\mathbb{C}} X$ for some subtorus S.
Since $\overline{S \cdot X} \supset X, S \cdot \mathbf{x}$, we must have $\dim_{\mathbb{C}} \overline{S \cdot X} = \dim_{\mathbb{C}} S = \dim_{\mathbb{C}} X$, but this forces everything to be equal.