# BOUNDS FOR TORSION HOMOLOGY OF ARITHMETIC GROUPS

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Abstract. This note presents as a survey some ideas used in my paper "Torsion homology of arithmetic lattices and $K_2$ of imaginary fields". It is essentially based on my talk at the conference "Growth and Mahler measures in geometry and topology", Djursholm, July 2013.

## 1. Introduction

Let $X = G/K$ be a symmetric space of noncompact type, i.e., $G$ is a semisimple Lie group without compact or Euclidean factors and $K \subset G$ is a maximal compact subgroup. For example, the case $G = \mathrm{PSL}_2(\mathbb{C})$ corresponds to $X = \mathbf{H}^3$, the hyperbolic space of dimension 3. A *lattice* $\Gamma \subset G$ is a discrete subgroup of finite covolume in $G$ (with respect to any Haar measure on $G$). It defines an *orbifold* $\Gamma\backslash X$, which has finite volume with respect to the $G$-invariant Riemannian metric on $X$. We say that $\Gamma$ is *uniform* if the quotient $\Gamma\backslash X$ is compact. If $\Gamma$ is torsion-free then the quotient $M = \Gamma\backslash X$ is a manifold, locally isometric to $X$. By Selberg's lemma, any quotient orbifold of finite volume is covered with finite degree by a manifold. The study of lattices of $G$ (or orbifolds) can be reduced to *irreducible* ones, i.e., those which are not commensurable with a product of two subgroups that are lattices in factors of $G$. If $G$ is simple, then trivially all lattices are irreducible.

According to a theorem of Gromov, the Betti numbers of the locally symmetric spaces $M = \Gamma\backslash X$ can be bounded linearly in the volume $\mathrm{vol}(M)$. This shows that the complexity of the topology of the manifold $M$ is controlled by a simple geometric invariant, the volume. Gromov's theorem is valid in a more generic context than for symmetric spaces, the relevant condition being the negative curvature. Moreover, it has been recently extended to the case of orbifolds by Samet [6]. For a fixed semisimple Lie group $G$ as above (of noncompact type, without compact factors and without center), the result can be formulated as the following. Here $b_j(\Gamma)$ denotes the rank of the group homology $H_j(\Gamma)$ (with coefficient in $\mathbb{Z}$) of $\Gamma$.

**Theorem 1** (Samet)**.** *There exists a constant $C_G$ such that for any irreducible lattice $\Gamma \subset G$ we have*

$$b_j(\Gamma) \leq C_G \, \mathrm{vol}(\Gamma\backslash X),$$

*for any $j$.*

A natural question is to ask whether a similar result can be proved for the torsion part of the homology $H_j(\Gamma)$. A motivation is the growing interest in torsion homology of arithmetic lattices due to connection with number theory; see [2, 3]. An *arithmetic group* is a group of the form (or more precisely commensurable to) $H(\mathbb{Z})$ where $H$ is an algebraic $\mathbb{Q}$-group. By a theorem of Borel and Harish-Chandra, such a group is always a lattice in $H(\mathbb{R})$ provided the latter is semisimple. Moreover, Margulis proved as a consequence of its superrigity theorem that every irreducible lattice in a semisimple Lie group of real rank at least 2 is arithmetic.

This note presents a short introduction to some ideas used in [4] to obtain upper bounds on the torsion homology of arithmetic lattices; here we do not discuss the aspects related to $K_2$ though. It is essentially based on my talk at the conference "Growth and Mahler measures in geometry and topology", Djursholm, July 2013. I would like to thank again both organizers, Eriko Hironaka and Ruth Kellerhals, for having invited me to this very nice workshop. I also thank the staff of the Mittag-Leffler institute for the perfect organization.

## 2. Consequences of Lehmer's conjecture

Lehmer's conjecture asserts that the Mahler measure of a noncyclotomic polynomial with integral coefficients is bounded away from 1. A well-known consequence of this conjecture is the following (cf. [5, Section 10]).

**Conjecture 2** (Short geodesic conjecture)**.** *Let $X$ be a symmetric space of noncompact type. There exists $\epsilon > 0$ such that the length of any closed geodesic on an arithmetic locally symmetric space $M = \Gamma\backslash X$ is greater than $\epsilon$.*

Let us restrict ourselves to compact manifolds $M = \Gamma\backslash X$. Assuming Conjecture 2 we can embed around any $x \in M$ a geodesic ball $B_r(x)$ of radius $r = \epsilon/2$. One needs about $\mathrm{vol}(M)/\mathrm{vol}(B_r(x))$ balls to cover all $M$. Using the notion of *nerve* of an open covering like this, one can show that $M$ is homotop to simplicial complex whose size (i.e, the total number of simplices) is bounded linearly in $\mathrm{vol}(M)$. Thus, the following statement is a consequence of Conjecture 2.

**Conjecture 3.** *There exists a constant $\beta_X$ such that any compact arithmetic manifold $M = \Gamma\backslash X$ is homotop to a simplicial complex of size bounded above by $\beta_X \mathrm{vol}(M)$.*

A statement like this is exactly what we need to bound the torsion homology of $M$. In general, in a complex of abelian groups of bounded ranks the torsion homology does not need to be bounded (take $\mathbb{Z} \xrightarrow{N} \mathbb{Z}$ with $N$ arbitrarily large). However, in a simplicial complex the "boundary maps" are concretely given by the boundary of simplices. This observation can be used to bound the torsion homology. More precisely, we can use the following result of Gabber to do so. For a proof see [7, §2.1]. For an abelian group $A$, we denote by $A_{\mathrm{tors}}$ its subgroup of torsion elements.

**Lemma 4** (Gabber)**.** *Let $A = \mathbb{Z}^a$ with the standard basis $(e_i)_{i=1,\dots,a}$ and $B = \mathbb{Z}^b$, so that $B \otimes \mathbb{R}$ is equipped with the standard Euclidean norm $\|\cdot\|$. Let $\phi : A \to B$ be a $\mathbb{Z}$-linear map such that $\|\phi(e_i)\| \le \alpha$ for each $i = 1, \dots, a$. If we denote by $Q$ the cokernel of $\phi$, then*

$$|Q_{\mathrm{tors}}| \le \alpha^{\min\{a,b\}} .$$

Thus, applying Lemma 4 to Conjecture 3 we would obtain a bound, for any $j$,

$$\log |H_j(M)_{\mathrm{tors}}| \le C_X \, \mathrm{vol}(M) \tag{2.1}$$

for some constant $C_X$.

**Remark 5.** For any $n \geq 2$, Conjecture 2 fails for *nonarithmetic* hyperbolic $n$-manifolds (see [1]). Moreover, Conjecture 3 fails for nonarithmetic hyperbolic 3-manifolds: the statement in the conjecture implies the finiteness of the number of manifolds of bounded volume, which is known to fail by Thurston-Jørgensen description of the volume spectrum of 3-manifolds. Note that restricted to the class of arithmetic manifolds (or lattices) this finiteness property holds for any symmetric space $X$, by a theorem of Borel and Prasad.

## 3. The case of noncompact manifolds

The upper bound for torsion homology (2.1) that would follow from Lehmer's conjecture in the case of compact arithmetic manifolds can actually be proved unconditionally for *noncompact* arithmetic manifolds. This is due to the classification of nonuniform arithmetic lattices, described in the next proposition (cf. [5, Lemma 5.2]).

**Proposition 6.** *Let $G$ be a semisimple real Lie group as in Section 1. If $\Gamma \subset G$ is an irreducible nonuniform arithmetic subgroup, then it can be written as a subgroup $\Gamma \subset H(\mathbb{Z})$ of finite index, where $H$ is an algebraic $\mathbb{Q}$-group of same (real) dimension as $G$.*

**Example 7.** Every nonuniform arithmetic lattice in $\mathrm{PSL}_2(\mathbb{C})$ is commensurable to a Bianchi group $\mathrm{PSL}_2(\mathcal{O}_F)$ for some imaginary quadratic field $F$. The corresponding $\mathbb{Q}$-group is then given by Weil's restriction of scalars as $H = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{PSL}_2/F)$, which has dimension $6 = \dim(\mathrm{PSL}_2)\cdot[F:\mathbb{Q}]$. To obtain all uniform lattices instead, one needs to take into account fields $F$ of (arbitrarily) large degrees, which will give raise to algebraic $\mathbb{Q}$-groups of larger dimensions.

As a consequence of Proposition 6, any element in a nonuniform arithmetic group $\Gamma \subset G$ can be written as a matrix whose characteristic polynomial has integral coefficients and fixed degree equal to $\dim(G)$. But Lehmer's conjecture is known to hold when the degrees of the polynomials is fixed (or bounded), and this will show that the length of closed geodesics is bounded away from 0. In other words:

**Proposition 8.** *Conjecture 2 holds for noncompact arithmetic manifolds $M = \Gamma\backslash X$.*

From this result, one can use the same argument to obtain a version of the statement of Conjecture 3 valid for noncompact manifolds. The problem is that the argument used there assumed that the manifolds were compact. To deal with noncompact manifolds one needs to remove some neighbourhoods of their unbounded part and check that the complexity of the topology does not become too bad where the cuts are performed. This is a difficult task, which could be done by Gelander in his thesis. He obtained the following result (see [5]):

**Theorem 9** (Gelander)**.** *There exists a constant $\beta_X$ such that any noncompact arithmetic manifold $M = \Gamma\backslash X$ is homotop to a simplicial complex of size bounded above by $\beta_X \mathrm{vol}(M)$.*

Together with Lemma 4 one then obtains the following.

**Corollary 10.** *There exists a constant $C_X$ such that for any noncompact arithmetic manifold $M = \Gamma\backslash X$, we have for any $j$:*

$$\log|H_j(M)_{\mathrm{tors}}| \leq C_X \,\mathrm{vol}(M).$$

## 4. An extension to the case of orbifolds

For arithmetic application especially, it is useful to have a version a Corollary 10 where the arithmetic subgroups $\Gamma \subset G$ may contain torsion, i.e., so that the quotient $\Gamma\backslash X$ is in general an orbifold. Such an extension was obtained in [4] for $G$ respecting some conditions, and it was applied to obtain upper bound for $K_2$ of the ring of integers of totally imaginary number fields.

**Theorem 11** (Emery)**.** *Let $G$ be a semisimple Lie group as in Section 1 and such that for any (nonuniform arithmetic) irreducible lattice $\Gamma_0 \subset G$ we have $H_q(\Gamma_0, \mathbb{Q}) = 0$ for $q = 1, \ldots, j$. Then, there exists a constant $C_G$ such that for any nonuniform arithmetic irreducible lattice $\Gamma \subset G$ we have:*

$$\log |H_j(\Gamma)_{\mathrm{tors}}| \leq C_G \operatorname{vol}(\Gamma\backslash G).$$

*Proof.* Using Proposition 6 one sees that we can construct for any $\Gamma$ as torsion-free normal subgroup $\Gamma_0 \subset G$ whose index is bounded by a constant depending only on $\dim(G)$. Then, the idea is to use Lyndon-Hochschild-Serre spectal sequence

$$E^2_{pq} = H_p(\Gamma/\Gamma_0, H_q(\Gamma_0)) \implies H_{p+q}(\Gamma),$$

together with Corollary 10, which gives an upper bound for $H_q(\Gamma_0)$. We refer to [4] for the details. □

For example, if $G$ has real rank at least 2 then superrigidity implies at once vanishing of the first Betti number and arithmeticity of lattices. Thus, we get the following result for torsion homology in degree one.

**Corollary 12.** *Let $G$ be a semisimple real Lie group without compact factor and of real rank at least 2. Then, there exists a constant $C_G$ such that for any nonuniform irreducible lattice $\Gamma \subset G$ we have:*

$$\log |H_1(\Gamma)_{\mathrm{tors}}| \leq C_G \operatorname{vol}(\Gamma\backslash G).$$

## References

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