Stabilisation in algebraic statistics

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Gaussian $k$-factor model

- $X_1, \ldots, X_n$ real, jointly Gaussian random variables, mean 0
- distribution determined by covariance matrix $\Sigma$
- suppose $n \gg k$ and $X_i = \sum_{j=1}^{k} \lambda_{ij} Y_j + \sigma_i Z_i$ where $Y_1, \ldots, Y_k, Z_1, \ldots, Z_n$ independent, standard normal factors noise

$\rightsquigarrow \Sigma = \Lambda \Lambda^T + \text{diag}(\sigma_i^2)$ rank $k$ plus diagonal

- $F_{k,n} = \{\text{all such matrices}\} \subseteq \mathbb{R}^{n \times n}$

**Question (DSS 07):** Generators for the ideal in $\mathbb{R}[(\sigma_{ij})]$ of $F_{k,n}$?

**Example**

1. $\sigma_{ij} - \sigma_{ji}$ and
2. off-diagonal $(k + 1) \times (k + 1)$-minors of $\Sigma$
3. Ideal of $F_{2,5}$ gen by 1. and $\frac{1}{10} \sum_{\pi \in S_5} \text{sgn}(\pi) \pi(\sigma_{12} \sigma_{23} \sigma_{34} \sigma_{45} \sigma_{15})$ pentad
Stabilisation?

\[ F_{k,n+1} \rightarrow F_{k,n} \] by forgetting last row and col, and \( F_{k,n} \) is \( S_n \)-stable

**Question (DSS)**
Is there \( n_0 = n_0(k) \) s.t. \( I(F_{k,n}) = \langle S_n I(F_{k,n_0}) \rangle \) for \( n \geq n_0 \)?

**Partial answer**
1. Yes for \( k = 1 \) with \( n_0 = 4 \) (2 × 2-minors).
2. Yes for \( k = 2 \) with \( n_0 = 6 \) (pentads and off-diagonal 3 × 3-minors generate \( I(F_{2,n}) \)) [B-D]
3. Yes topologically for each \( k \) (no idea about \( n_0 \)) [D]
**Definition** An FI-algebra over $K$ is a functor $A$ from FI to (commutative, associative, unital) $K$-algebras. An ideal is an FI-submodule of $A$ such that for each $S$, $I(S)$ is an ideal in $A(S)$.

**General question**

A an FI-algebra over $K$, $I \subseteq A$ an ideal of interest, is $I$ f.g.? (In $k$-factor model, $A(S) = \mathbb{R}[\sigma_{ij} | i, j \in S]$ and $I(S) = I(F_{k,S})$.)

*not Noetherian!*

FI-algebra $A \mapsto$ functor $X$ from $\text{FI}^{\text{op}}$ to $\text{Top}$, $S \mapsto \text{Hom}(A(S), K)$ with Zariski topology. Weaker question: is there a finite number of elements of $I(S)$’s whose ideal define $V(I) \subseteq X$?

The topological space defined by ideal $J \subseteq I(F_{k,\cdot})$ generated by the off-diagonal $(k + 1) \times (k + 1)$-minors is Noetherian.
Some results on FI-algebras

**Theorem** (C, A-H) The FI-algebra $R : S \mapsto K[S]$ is Noetherian.

**Theorem** (C, H-S) So are tensor powers $R^\otimes d$.

For any functor $Y$ from FI to sets, can form the FI-algebra $K[Y] : S \mapsto K[Y_S]$.

**Theorem** (D-E-K-L) If $Y$ is f.g. and $\phi : K[Y] \to R^\otimes k$ monomial, then $\ker \phi$ is f.g. and $\text{im} \phi$ is Noetherian.

Unfortunately, $F_{k,n}$ was not parameterised by monomials (except for $k = 1$, forgetting diagonal), so this does not help there. But it does elsewhere!
Markov random fields

\[ \Gamma = (V, E) \] finite, simple, undirected graph
for each \( v \) have a finite set \( S_v \) of states
for each \((\text{max})\) clique \( C \subseteq V \) have function \( \phi_C : \prod_{v \in C} S_v \to \mathbb{R}_+ \)
\[ \rightsquigarrow \] a probability distribution on \( \prod_{v \in V} S_v \) by \( P(i) = \prod_{C} \phi_C(i|C) \)

**Question**
What are the polynomial relations among the \( P(i) \) as the \( \phi_C \) vary?

**Examples**
\[ P(i, j, k) = a_i b_j c_k \] (independence, rank-one tensors)

ideal generated by quadratic equations, *independently of the* \( |S_v| \).

\[ K_{3,N} \ P(i_1, i_2, i_3, j_1, \ldots, j_n) = \prod_{l=1}^{N} a_{i_1,j_l} b_{i_2,j_l} c_{i_3,j_l} \]

**Theorem** (Rauh-Sullivant) for \( K_{3,N} \), if all \( |S_v| = 2 \), ideal generated in degree \( \leq 12 \), *independently of* \( N \).
The independent set theorem

**Theorem** (Hillar-Sullivant)
Let $U \subseteq V$ be an independent set. Then the relations among the $P(i)$ stabilise as $|S_v| \to \infty$ for $u \in U$, while the $S_v$ with $v \in V \setminus U$ remain fixed.

**Proof**

$\prod_{u \in U} S_u \mapsto K[p_i | i \in \prod_{v \in V} S_v]$ defines a f.g. FI$^U$-algebra

Consider the homomorphism $\phi : p_i \mapsto P(i) = \prod_C \phi_C(i|C)$

$$= \prod_{C \cap U = \emptyset} \phi_C(i|C) \cdot \prod_{u \in U} (\prod_{C \ni u} \phi_C(i|C)) \quad \text{(each } |C \cap U| \leq 1)$$

- constant stuff
- only one index unbounded

use the result about monomial maps. \qed
A second independent set theorem

**Theorem** (in progress, with Oosterhof, Rauh, Sullivant)
Let $U$ be an independent set. Now keep all state set sizes equal, but repeatedly clone vertices in $U$, along with their sets $S_u$. Then the relations among the $P(i)$ stabilise up to symmetry.

For simplicity $|U| = 1$. Denote by $W$ the set of clones, including the original vertex. (So $W$ is independent in the new graph.)

$$P(i) = \prod_{C \cap W = \emptyset} \phi_C(i|C) \cdot \prod_{w \in W} (\prod_{C \ni w} \phi_C(i|C)) \quad \text{(each } |C \cap W| \leq 1)$$

For each fixed value of $i|V \setminus W$ you see the entries of a rank-one tensor in $\mathbb{R}^W$. We’ll prove a theorem about tuples of rank-one tensors.
An $F S^{\text{op}}$-algebras

Recall
FS is the category of finite maps with surjections

**Theorem** (Sam-Snowden)
F.g. $F S^{\text{op}}$-modules are Noetherian

**Example of an $F S^{\text{op}}$-algebra**
fix $n \in \mathbb{N}$, then surjection $S \rightarrow S'$ gives injection $[n]^{S'} \rightarrow [n]^S$
and hence algebra injection $K[y_\alpha \mid \alpha \in [n]^{S'}] \rightarrow K[y_\alpha \mid \alpha \in [n]^S]$. This is a f.g. $F S^{\text{op}}$ algebra $T_n$ (for tensor), coord ring of $(K^n)^{\otimes S}$.

**Theorem** (Draisma-Kuttler)
For each fixed $r$, this algebra has a finitely generated $F S^{\text{op}}$-ideal whose zero set is the variety of border-rank-$r$ tensors.
Rank-one tensors

In $T_n$, let $I_n$ denote the ideal of the rank-one tensors.

**Theorem (D-O-R-S)**

$Q_n := T_n/I_n$ is a Noetherian $\text{OS}^{\text{op}}$-algebra; also for several copies.

**Recall**

OS has linearly ordered finite sets $[k]$ and morphisms $f: [k] \to [l]$ such that $i < j$ implies $\max f^{-1}(i) < \max f^{-1}(j)$

$Q_n([l])$ is the monoid algebra of the additive monoid of matrices in $(\mathbb{Z}_{\geq 0})^{n \times l}$ with constant column sum.

Let $A \in (\mathbb{Z}_{\geq 0})^{n \times l}$ and $B \in (\mathbb{Z}_{\geq 0})^{n \times k}$ be such matrices. Call $A \preceq B$ if there exists ordered surjective $f: [k] \to [l]$ such that $b_{ij} \geq a_{if(j)}$ for all $i, j$. 
Proposition
This is a w.p.o.

To each $A$ associate the monomial ideal $J_A$ in $K[x_1, \ldots, x_n]$ generated by the $x^\alpha$ with $\alpha$ running over the columns of $A$.

Theorem (Maclagan)
These monomial ideals are w.p.o. by reverse inclusion: in any sequence $J_1, J_2, \ldots \exists i < j : J_i \supseteq J_j$.

- Suppose there are bad sequences $A_1, A_2, \ldots$.
- Then these exist such that $J_{A_1} \supseteq J_{A_2} \supseteq \ldots$ (*).
- Take a minimal such bad sequence and write $A_i = (a_i|B_i)$.
- Find $i_1 < i_2 < \ldots$ such that $a_{i_1} \leq a_{i_2} \leq \ldots$ and moreover $J_{B_{i_1}} \supseteq J_{B_{i_2}} \supseteq \ldots$
Claim: $A_1, \ldots, A_{i_1 - 1}, B_{i_1}, B_{i_2}, \ldots$ is smaller bad sequence.

• suppose $A_i \leq B_j$, witnessed by $f : \lfloor l_j - 1 \rfloor \to \lfloor l_i \rfloor$. Then $A_i \leq A_j$, witnessed by $g : \lfloor l_j \rfloor \to \lfloor l_i \rfloor$ with $g(m) = f(m - 1)$, $m > 1$ and $g(1) = r$ such that $r$-th column of $A_i \leq$ first column of $A_j$ (exists by (*)).

• suppose that $B_i \leq B_j$, witnessed by $f : \lfloor l_j - 1 \rfloor \to \lfloor l_i - 1 \rfloor$. Then $A_i \leq A_j$ witnessed by $g : \lfloor l_j \rfloor \to \lfloor l_i \rfloor$ defined by $g(m) = f(m - 1) + 1$ and $g(1) = 1$. □

The ideals in the second independent set theorem are $\text{FS}^{\text{op}}$-ideals in $T_n$ (actually, a tensor product of copies), hence f.g.

There are many more graphical models, e.g. from phylogenetics, where stabilisation occurs!
Nonnegative rank

**Definition**

$M \in \mathbb{R}_{\geq 0}^{m \times n}$ has *nonnegative rank* $\leq r$ if $M = A \cdot B$ with $A \in \mathbb{R}_{\geq 0}^{m \times r}$ and $B \in \mathbb{R}_{\geq 0}^{r \times k}$.

Consider the boundary $B_r^{m \times n}$ of the set of matrices of nonnegative rank $\leq r$ in the variety of matrices of rank $\leq r$.

**Theorem** (Mond-Smith-v. Straten) for $r = 3$ this has 3 $S_m \times S_n$-orbits of components, independent of $m, n$.

(Quantifier-free description by Kubjas-Robeva-Sturmfels, and ideals by Eggermont-Horobet-Kubjas.)

Higher rank $r$? Ongoing work by Horobet-Chen.