Matroids: algebraicity, duality, and valuations

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Algebraic matroids and Frobenius flocks, Bollen-D-Pendavingh
Matroids over one-dimensional groups, Bollen-Cartwright-D
Recipe: Given an $n \times d$-matrix over a field, remember only which subsets of the rows are independent.
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From matrix to matroid I

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(char ≠ 2)

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(non-)Fano matroid
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This collection \( \mathcal{I} \subseteq 2^{[n]} \) is nonempty, downward closed, and satisfies \( \forall I, J \in \mathcal{I} : |J| > |I| \Rightarrow \exists j \in J \setminus I : I + j \in \mathcal{I} \); these are the defining properties of a matroid on \([n]\).
Well-understood breeds of matroids

**Definition:** A matroid on \([n]\) is a nonempty, downward closed collection \(\mathcal{I} \subseteq 2^{[n]}\) s.t. \(\forall I, J \in \mathcal{I} : |J| > |I| \Rightarrow \exists j \in J \setminus I : I + j \in \mathcal{I}\). The maximal independent sets are called the bases.
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**Linear matroids:** from a matrix over a field.

**Graphical matroids:** edge set \([n]\), independent = contains no cycle.

A basis:

The greedy algorithm for minimal-cost spanning tree carries over precisely to matroids.
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**Definition:** A *matroid* on \([n]\) is a nonempty, downward closed collection \(\mathcal{I} \subseteq 2^{[n]}\) s.t. \(\forall I, J \in \mathcal{I} : |J| > |I| \implies \exists j \in J \setminus I : I + j \in \mathcal{I}\). The maximal *independent sets* are called the *bases*.

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The greedy algorithm for minimal-cost spanning tree carries over precisely to matroids.

*Every graphical matroid is linear (over every field).*
Definition: Let $L \supseteq K$ be a field extension and $x_1, \ldots, x_n \in L$. Set $\mathcal{I} := \{I \subseteq [n] : (x_i)_{i \in I} \text{ algebraically independent over } K\}$. Such a matroid is called algebraic (over $K$).
The ugly ducks among matroids

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\]

\[
x_1 = t_1 \\
x_2 = t_2 \\
x_3 = t_3 \\
x_4 = t_2 + t_3 \\
x_5 = t_1 + t_3 \\
x_6 = t_1 + t_2 \\
x_7 = t_1 + t_2 + t_3
\]

$L = K(t_1, t_2, t_3)$
Why study algebraic matroids?

**Generic completion**

$K$ algebraically closed

$X \subseteq K^n$ irreducible closed subvariety

$\mathcal{I} := \{I \subseteq [n] : \text{any generic } p \in K^I \text{ can be completed to } \tilde{p} \in X\}$
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**Problem:** Given *X* and *I* ⊆ [n], decide whether *I* ∈ *I*.

Can be solved by Buchberger’s algorithm for elimination, but this is not efficient.
Case study: generic low-rank matrix completion

\[ [n] = [\ell] \times [m], \quad \ell, m \geq k, \quad K^{\ell \times m} \supseteq X := \{ A | \text{rk}(A) \leq k \} \]

**Generic rank-\(k\) completion problem:** On input \(I \subseteq [\ell] \times [m]\), decide whether a generic choice of \((a_{ij})_{(i,j) \in I}\) can be completed to a matrix of rank \(\leq k\).
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**Rank \(k = 1\):** yes if and only if the bipartite graph with edges \(I\) has no cycles \(\iff I\) is the graphical matroid of \(K_{\ell,m}\); independence is easy.
[n] = [ℓ] × [m], ℓ, m ≥ k, K\( \ell \times m \) ⊇ X := \{A | \text{rk}(A) ≤ k\}

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**Open problem:** is there a poly time deterministic algorithm that on input \(S \subseteq \mathbb{Q}^n\) decides if \(S\) can be partitioned by a hyperplane into two linearly independent sets?
Linearising algebraic matroids

$X \subseteq K^n$ irreducible, and $q \in X$ smooth $\leadsto$ the tangent space $T_qX$ defines a matroid on $[n]$ with $\mathcal{I}(T_qX) \subseteq \mathcal{I}(X)$. 
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If $\text{char} K = 0$, then for $q \in X$ sufficiently general, $\mathcal{I}(T_qX) = \mathcal{I}(X)$; not true for $\text{char} K = p > 0$.

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\[ X \subseteq K^n \text{ irreducible, and } q \in X \text{ smooth} \Rightarrow \text{the tangent space } T_qX \text{ defines a matroid on } [n] \text{ with } \mathcal{I}(T_qX) \subseteq \mathcal{I}(X). \]

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Consequences
• Algebraic matroids in characteristic 0 are linear (Ingleton)
• Sometimes there is an efficient probabilistic algorithm for the generic completion problem: sample \( q \in X \), compute \( T_qX \), and use Gaussian elimination to check \( I \in \mathcal{I}(T_qX) \).
Duality

**Definition:** If $\mathcal{I}$ is a matroid on $[n]$, then $\mathcal{I}^\perp := \{J \subseteq [n] : J \text{ is disjoint from some basis of } \mathcal{I}\}$ is the *dual* matroid.
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The dual of a *linear* matroid is linear:

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A = \begin{bmatrix}
1 & 0 & 0 \\
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0 & 0 & 1 \\
0 & 1 & 1 \\
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1 & 1 & 0 \\
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\end{bmatrix}
\quad \xrightarrow{\text{char} \neq 2} \quad A^\perp = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 \\
-1 & -1 & 1 & 1 \\
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The dual of a *planar graph* matroid is graphical:
Main challenges on algebraic matroids

**Testing algebraicity:** given a matroid, how does one decide if it is algebraic?

For *linearity*, this boils down to testing whether a system of polynomial equations has a solution, and Buchberger’s algorithm can do this.
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Testing algebraicity: given a matroid, how does one decide if it is algebraic?

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Duality: is the dual of an algebraic matroid again algebraic?
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Yes in characteristic 0, because they’re linear!
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Yes in characteristic 0, because they’re linear!

Example (Alfter-Hochstättler): the tic-tac-toe matroid on $[3] \times [3]$ has as bases all quintuples except all 4 L’s and all 4 T’s. Is it algebraic?? Its dual is not.
$K$ a field, $v : K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ a non-Archimedean valuation: $v^{-1}(\infty) = \{0\}$, $v(ab) = v(a) + v(b)$, and $v(a + b) \geq \min\{v(a), v(b)\}$.
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**Recipe (Dress-Wenzel):** Given an $n \times d$-matrix of rank $d$ over $K$, remember the valuations of the $d \times d$-subdeterminants.
From matrix to matroid (valuation) II

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$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 8 \end{bmatrix}$$

$K = \mathbb{Q}$, $v$ = 2-adic

$$\begin{align*}
\mu(\{1,2\}) &= \mu(\{1,3\}) \\
&= \mu(\{2,3\}) = \mu(\{2,4\}) \\
&= \mu(\{3,4\}) = 0 \\
\mu(\{1,4\}) &= 3
\end{align*}$$
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$$

$\mu(\{1, 4\}) = 3$

This matroid valuation $\mu : \binom{[n]}{d} \to \bar{\mathbb{R}}$ satisfies: $\mu \neq \infty$ and

$$
\forall B, B', i \in B \setminus B' \exists j \in B' \setminus B : \mu(B) + \mu(B') \geq \mu(B - i + j) + \mu(B' + i - j).
$$

Matroid valuations play the role of linear spaces in trop geometry.
Definition (Bollen-D-Pendavingh, Cartwright)

Let $K$ be an algebraically closed field of characteristic $p > 0$, and let $L = K(x_1, \ldots, x_n) \supseteq K$ be a field extension of transcendence degree $d$. Define a map $\mu : \left( \binom{[n]}{d} \right) \to \overline{\mathbb{R}}$ as

$$\mu(I) := \log_p [L : K((x_i)_{i \in I})^{sep}]$$

This map is the Lindström valuation of the algebraic matroid.
The Lindström valuation

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$\sim \mu : \begin{pmatrix} [n] \\ d \end{pmatrix} \rightarrow \overline{\mathbb{R}}$ defined as $\mu(I) := \log_p[L : K((x_i)_{i \in I})^{\text{sep}}]$ is the Lindström valuation of the algebraic matroid.

Theorem (B-D-P): if the Lindström valuation is trivial, i.e. $\exists \alpha \in \mathbb{Z}^n$: for all bases $\mu(B) = \sum_{i \in B} \alpha_i$, then the algebraic matroid is also linear.
The Lindström valuation

Definition (Bollen-D-Pendavingh, Cartwright)

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$$\varphi \colon \binom{[n]}{d} \to \overline{\mathbb{R}}$$

is defined as

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Theorem (B-D-P): if the Lindström valuation is trivial, i.e. $\exists \alpha \in \mathbb{Z}^n$: for all bases $\mu(B) = \sum_{i \in B} \alpha_i$, then the algebraic matroid is also linear.

Corollary: Matroids, such as Fano, that admit only trivial valuations are algebraic over $K$ iff they are linear over $K$.

Bollen used enhancements of this for ruling out algebraicity of many matroids on $\leq 9$ elements.
$K$ algebraically closed, char$K = p > 0$
G a one-dimensional algebraic group over $K$
⇝ then $G = (K, +)$ or $G = (K^*, \cdot)$ or $G = \text{an elliptic curve.}$
Matroids over one-dimensional groups

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$G$ a one-dimensional algebraic group over $K$  
$\leadsto$ then $G = (K, +)$ or $G = (K^*, \cdot)$ or $G = \text{an elliptic curve}$.

Construction: a closed, connected subgroup $X \subseteq G^n$  
$I := \{I \subseteq [n] : X \to G^I \text{ is surjective} \}$ is an algebraic matroid.

Questions: Lindström valuation? Is the dual also algebraic?
Matroids over one-dimensional groups

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**Key to the solution:** the endomorphism ring \( \mathbb{E} \) of \( G \):
• \( K[F] \) with \( Fa = a^pF \) if \( G = (K,+); \)
• \( \mathbb{Z} \) if \( G = (K^*,\cdot) \); and
• \( \mathbb{Z} \) or an order in an imaginary quadratic number field or in a quaternion algebra if \( G = \text{an elliptic curve} \).

In all cases, \( \mathbb{E} \) is an Ore ring, hence generates a skew field \( \mathbb{Q} \).
The closed subgroup $X \subseteq G^n$ is the image of a map $G^d \to G^n$ given by a rank-$d$ matrix $A \in \mathbb{F}^{n \times d}$, and is uniquely determined by the column span of $A$, a right $Q$-subspace of $Q^n$. 
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The ring $\mathbb{E}$ comes with a valuation: $v(\alpha)$ is the degree of inseparability of $\alpha : G \to G$; this extends to $v : Q \to \overline{\mathbb{R}}$. 
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The ring \( \mathbb{E} \) comes with a valuation: \( v(\alpha) \) is the degree of inseparability of \( \alpha : G \to G \); this extends to \( v : \mathbb{Q} \to \mathbb{R} \).

**Theorem (B-Cartwright-D)**
The Lindström valuation of the matroid defined by \( X \) maps \( I \subseteq [n] \) of size \( d \) to \( v(\text{Diedonné determinant of } A[I]) \).
From matrix to matroid (valuation) III

The closed subgroup $X \subseteq G^n$ is the image of a map $G^d \to G^n$ given by a rank-$d$ matrix $A \in \mathbb{E}^{n \times d}$, and is uniquely determined by the column span of $A$, a right $Q$-subspace of $Q^n$.

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The Lindström valuation of the matroid defined by $X$ maps $I \subseteq [n]$ of size $d$ to $v(\text{Diedonné determinant of } A[I])$.

**Theorem (B-C-D)**
The dual matroid is also that of a closed subgroup $X^\vee$ of $G^n$.

$\text{Colspace}(A)^\perp$ is a left subspace, but fortunately $Q \cong Q^{\text{op}}$. 
Dual valuations

Definition (Dress-Wenzel)
If $\mu : \left( \begin{bmatrix} n \\ d \end{bmatrix} \right) \rightarrow \bar{\mathbb{R}}$ is a valuation, then $\mu' : \left( \begin{bmatrix} n \\ n - d \end{bmatrix} \right) \rightarrow \bar{\mathbb{R}}$, $\mu'(I) = \mu(I^c)$ is the dual valuation.
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This notion is compatible with the dual of a linear matroid, but *not* with the construction of $X'$ above: take $G = (K, +)$, $E = K[F]$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & F \end{bmatrix} \quad \Rightarrow \quad A^\perp = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & F & 0 & -1 \end{bmatrix} \quad \Rightarrow \quad A^\vee = \begin{bmatrix} 1 & 1 \\ 1 & F^{-1} \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$
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\end{bmatrix}
\]

\( \mu(14) + \mu(23) - \mu(13) - \mu(24) = 1 + 0 - 0 - 0 = 1 \) but \( \mu^\vee(23) + \mu^\vee(14) - \mu^\vee(24) - \mu^\vee(13) = -1 + 0 - 0 - 0 = -1 \)
Theorem (B-C-D): The set of Lindström valuations of algebraic matroids is not closed under duality.

Proof sketch: via a universality construction of Evans-Hrushovski, we construct a matroid $M^\vee$ s.t. every algebraic realisation of $M^\vee$ is equivalent to one from a subgroup $X^\vee \subseteq G^n$ for some one-dimensional algebraic group $G$, but such that the Lindström valuation of $X$ is not the dual to that of $X^\vee$. Then the dual of the Lindström valuation of $X$ is not a Lindström valuation.
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Summarising

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Thank you!