Constructing Lie algebras from extremal elements

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Eindhoven, 18 March 2010
Sandwich and extremal elements

$L$ Lie algebra over $K$, char $K \neq 2$

$x \in L$ is sandwich if $xxL := [x, [x, L]] \subseteq Kx$

$x \in L$ is extremal if $xxL \subseteq Kx$

$\iff \exists f_x : L \to K$ linear : $xxy = f_x(y)x$

Goal: analyse “moduli spaces” of Lie algebras with distinguished extremal generators.
Sandwich algebras

Γ = (Π, E) finite, undirected graph
K ground field of characteristic ≠ 2
F = (free Lie algebra over K generated by Π)/(xy for x, y ∈ Π, x ∼ y)
I₀ ⊆ F generated by xxy, x ∈ Π, y ∈ F
S := F/I₀

Theorem (Kostrikin-Zelmanov) S is finite-dimensional.

V graded, finite-dimensional with F = V ⊕ I₀
Note: V ⊇ KΠ

Example
Γ = K₃
V = ⟨x, y, z, xy, xz, yz, xyz, yzx⟩ₖ
The moduli space

\[ f = (f_x)_{x \in \Pi} \in (F^*)^\Pi \]
\[ \mapsto I(f) \subseteq F \text{ generated by } xxy - f_x(y)x, \ x \in \Pi, \ y \in F \]
\[ L(f) := F/I(f) \]
\[ \mapsto I(0) = I_0 \text{ and } S = L(0) \]

Proposition (Cohen-Steinbach-Ushirobira-Wales)
\[ V + I(f) = F \text{ for all } f. \]

**Definition**
\[ \mathcal{M}(\Gamma) := \{ f \in (F^*)^\Pi | \dim L(f) = \dim S \} = \{ f \in (F^*)^\Pi | F = V \oplus I(f) \} \]

moduli space of maximal-dimensional Lie algebras with extremal generators \( \leftrightarrow \Pi \)
commutation relations \( \leftrightarrow E^c \)
Finiteness

Theorem (D, in ’t panhuis, Postma, Roozemond)

\( \mathcal{M}(\Gamma) \) is naturally an affine \( K \)-variety of finite type.

Idea:

1. \( \mathcal{M}(\Gamma) \rightarrow (V^*)^\Pi, \)
   \( (f_x)_{x\in\Pi} \mapsto f|_V := (f_x|_V)_{x\in\Pi} \) injective
   (can express each \( f_x(y), \ x \in X, \ y \in F \) polynomially in \( f|_V \))

2. image given by polynomial equations

What does \( \mathcal{M}(\Gamma) \) look like?

What Lie algebras does it parameterise?
Example (Cohen-Steinbach-Ushirobira-Wales)

\[ \Gamma = K_3 \]
\[ \mathcal{M}(\Gamma) = \mathbb{A}^4_K \text{ with coordinates } f_x(y), f_y(z), f_z(x), f_x(yz) \]

Possible Lie algebras:
1. \( \mathfrak{sl}_3 \)
2. \( \mathfrak{sl}_2 \ltimes K^2 \oplus K^2 \oplus K^1 \)
3. \( \mathfrak{sl}_2 \ltimes K^2 \oplus K^2 \oplus K^1 \)
4. \( \mathfrak{n}_+ \ltimes \mathfrak{g}/\mathfrak{n}_+ \)
\[ \Gamma = \text{simply laced affine-type Dynkin diagram} \]

\[ \begin{array}{c}
\text{Theorem (D-in 't panhuis)}
\end{array} \]

\[ \mathcal{M}(\Gamma) \cong A_{|E|+1}^{|E|+1} \quad \text{and} \quad f \in \text{open dense subset} \Rightarrow L(f) \cong g \text{ Chevalley of type } \Gamma \]
Proof idea

Γ simply laced Dynkin diagram

1. determine $S$ using $\text{Kac-Moody}(\Gamma)$
   $\rightsquigarrow S \cong n_+ \ltimes g/n_+$
   $\rightsquigarrow V \mathbb{N}^{\Pi}$-graded

2. express $f|_V$ polynomially in $f_x(y)(= f_y(x)), x \sim y$
   and $f_{x_0}(m)$ for $m \in V$ of highest weight in $\mu \in \mathbb{N}^{\Pi}$ with $\mu_{x_0} = 0$
   $\rightsquigarrow \mathcal{M}(\Gamma) \subseteq \mathbb{A}^{\lvert E \rvert + 1}$ closed

3. find a sufficiently general point $f \in \mathcal{M}(\Gamma)$
   with $L(f)$ Chevalley of type $\Gamma$

4. move the point around in an open dense subset of $\mathbb{A}^{\lvert E \rvert + 1}$

Remark: over $\mathbb{C}$, simple Lie algebras are rigid.
The scaling torus

\[ T = (K^*)^\Pi \text{ acts on } M(\Gamma) \]

characters on \( \mathbb{A}^{|E|+1} \) for \( \Gamma \) affine Dynkin:
\[ \alpha_e := \alpha_x + \alpha_y, \ e = \{x, y\} \in E \]
\[ \delta := \alpha_{x_0} + \text{highest root} \]
(actually, \( -\alpha_e, -\delta \))

1. \( D_{\text{even}}^{(1)}, E_7^{(1)}, E_8^{(1)} \): independent characters
2. \( A_{\text{even}}, D_{\text{odd}}^{(1)}, E_6^{(1)} : \alpha_e, \ e \in E \) independent, \( \delta \) in their \( \mathbb{Q} \)-span
3. \( A_{\text{odd}}^{(1)} : \alpha_e, \ e \in E \) dependent and \( \delta \) in their \( \mathbb{Q} \)-span.
What about

1. other graphs? (in ’t panhuis-Postma-Roozemond)
2. equations for the complement of the open dense set?
3. recognition?
4. structure of $\mathcal{M}(\Gamma)$ in general?