(Uniform) determinantal representations

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$R := \mathbb{C}[x_1, \ldots, x_n]$ and $R_{\leq d} := \{ p \in R \mid \deg p \leq d \}$

**Definition**

A **determinantal representation** of $p \in R$ of size $N$ is a matrix $M \in R_{\leq 1}^{N \times N}$ with $\det(M) = p$. 
Determinantal representations

\[ R := \mathbb{C}[x_1, \ldots, x_n] \text{ and } R_{\leq d} := \{ p \in R \mid \deg p \leq d \} \]

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\( n = 1 \): companion matrices

\[
\begin{bmatrix}
  x & -1 \\
  \quad & x & -1 \\
  \quad & \quad & \quad & \ddots & \ddots \\
  \quad & \quad & \quad & \quad & x & -1 \\
  \quad & \quad & \quad & \quad & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} + a_n x
\end{bmatrix}
\]

\[
\det \left( \begin{bmatrix}
  x & -1 \\
  \quad & x & -1 \\
  \quad & \quad & \quad & \ddots & \ddots \\
  \quad & \quad & \quad & \quad & x & -1 \\
  \quad & \quad & \quad & \quad & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} + a_n x
\end{bmatrix} \right) = a_0 + a_1 x + \ldots + a_n x^n
\]
Determinantal representations

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**A bivariate example**

\[
\begin{vmatrix}
  x & -1 \\
  y & -1 \\
  a + bx + cy & dx + ey & fy
\end{vmatrix} = a + bx + cy + dx^2 + exy + fy^2
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Determinantal representations

\[ R := \mathbb{C}[x_1, \ldots, x_n] \text{ and } R_{\leq d} := \{ p \in R \mid \text{deg } p \leq d \} \]

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Determinantal representations always exist, but how small?

\( \leadsto \) the determinantal complexity \( \text{dc}(p) \) is the smallest \( N \).
Why?
Motivation I: permanent versus determinant

“If $p$ has a determinantal representation $M$ of small size $N$, then $p$ can be evaluated efficiently using Gaussian elimination.”
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**Definition**

$$\text{perm}_m := \sum_{\pi \in S_m} x_{1\pi(1)} \cdots x_{m\pi(m)}$$ is the $m \times m$ permanent.

**Example**

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 3 \text{ counts perfect matchings: }$$

\begin{center}
\includegraphics[width=0.5\textwidth]{perfect_matchings.png}
\end{center}
Motivation I: permanent versus determinant

“If \( p \) has a determinantal representation \( M \) of small size \( N \), then \( p \) can be evaluated efficiently using Gaussian elimination.”

Definition
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\]

Example
\[
\text{perm}_3 \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 
\end{bmatrix} = 3 \text{ counts } \text{perfect matchings}:
\]

Counting matchings in bipartite graphs is believed hard, so \( \text{dc} (\text{perm}_m) \) should be large!
Conjecture

$\text{dc}(\text{perm}_m)$ grows faster with $m$ than any polynomial.
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Best known bounds

$\frac{m^2}{2} \leq dc(\text{perm}_m) \leq 2^m - 1$

[Valiant, 70s]

[Mignon-Ressayre 04, Grenet 12]

[Alper-Bogart-Velasco 15: = 7 for $m = 3$]
Geometric complexity theory

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dc(perm$_m$) grows faster with $m$ than any polynomial.

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\[
\frac{m^2}{2} \leq dc(\text{perm}_m) \leq 2^m - 1
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[Alper-Bogart-Velasco 15: $= 7$ for $m = 3$]

Proof sketch of lower bound
If $\psi : \mathbb{C}^{m \times m} \rightarrow \mathbb{C}^{N \times N}$ affine-linear with $\det_N(\psi(A)) = \text{perm}_m(A)$,
\[
J := \begin{bmatrix}
-m + 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}
\]
\[
\psi(J) = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}
\]
\[
\text{perm}_m = 0, \quad \det_N = 0
\]
Conjecture

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$q_1(X) := \text{quadratic part of } \text{perm}_m(J + X), \text{ form of rank } m^2$

$q_2(Y) := \text{quadratic part of } \det_N(\psi(J) + Y), \text{ form of rank } \leq 2N$
Geometric complexity theory

**Conjecture** [Valiant, 70s]
dc($\text{perm}_m$) grows faster with $m$ than any polynomial.

**Best known bounds** [Mignon-Ressayre 04, Grenet 12]
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\frac{m^2}{2} \leq \text{dc}(\text{perm}_m) \leq 2^m - 1 \quad \text{[Alper-Bogart-Velasco 15: } = 7 \text{ for } m = 3]\]

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Now $q_1(X) = q_2(L(X))$ where $L$ linear part of $\psi$, so $m^2 \leq 2N$. \qed
Grenet’s $2^m - 1$ construction

$x_{ij}$ labels an arrow from an $(i - 1)$-set to an $i$-set by adding $j$. 
Theorem [Landsberg-Ressayre, 15]
Grenet’s representation is optimal among representations that preserve left multiplication with permutation and diagonal matrices.
Geometric complexity theory

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**GCT Programme** [Mumuley-Sohoni, 01-]
Compare orbit closures $X_1, X_2$ of $\ell^{N-m}\text{perm}_m$ and $\det_N$ inside the space of degree-$N$ polynomials in $N^2$ variables under $G = \text{GL}_{N^2}$; try to show that $X_1 \not\subseteq X_2$ by showing that multiplicities of certain $G$-representations are higher in $\mathbb{C}[X_1]$ than in $\mathbb{C}[X_2]$ unless $N$ is super-polynomial in $m$. 
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**Theorem** [Bürgisser-Ikenmeyer-Panova, 16]
This approach does not work if higher than is restricted to $1 > 0$ (so-called occurrence obstructions).
Motivation II: Solving systems of equations

In numerics, solving a univariate equation \( p(x) = 0 \) is often done by finding the eigenvalues of the companion matrix of \( p \).
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Proposal [Plestenjak-Hochstenbach, 16]

To solve $p(x, y) = q(x, y) = 0$ write $p = \det(A_0 + xA_1 + yA_2)$ and $q = \det(B_0 + xB_1 x + yB_2)$ and solve the two-parameter eigenvalue problem $(A_0 + xA_1 + yA_2)u = 0$ and $(B_0 + xB_1 + yB_2)v = 0$. 
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$\Rightarrow$ translates to a joint pair of generalised eigenvalue problems:

$(\Delta_1 - x\Delta_0)w = 0$ and $(\Delta_2 - y\Delta_0)w = 0$ where $w = u \otimes v$ and

$\Delta_0 = A_1 \otimes B_2 - A_2 \otimes B_1, \Delta_1 = A_2 \otimes B_0 - A_0 \otimes B_2, \Delta_2 = A_0 \otimes B_1 - A_1 \otimes B_0$
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If the sizes are $N$, then $\Delta_i$ have size $N^2$, and solving takes $(N^2)^3 \ldots$ (plane curves have det rep of size = deg, but harder to compute).
Theorem [Boralevi-v Doornmalen-D-Hochstenbach-Plestenjak, 16]
For $n$ fixed, there exist $C_1, C_2$ such that a sufficiently general $p \in \mathbb{R}_{\leq d}$ has $dc(p) \geq C_1 d^{n/2}$ and any $p \in \mathbb{R}_{\leq d}$ has $dc(p) \leq C_2 d^{n/2}$. 
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For the upper bound, the determinantal representation can be chosen to depend bi-affine-linearly on $x_1, \ldots, x_n$ and on the coefficients of $p$; these are uniform determinantal representations.
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Proof of lower bound
If sufficiently general $p \in R_{\leq d}$ have $dc(p) \leq N$, then the map $\det : R_{\leq 1}^{N \times N} \to R_{\leq N}$ contains $R_{\leq d}$ in the closure of its image. Comparing dimensions, find $N^2 \cdot (n + 1) \geq \dim_{\mathbb{C}} R_{\leq d} = \binom{n + d}{n}$. □
**Definition**

Given a nonzero subspace $V \subseteq R$ write $V_{\leq d} := V \cap R_{\leq d}$. $V$ is connected to 1 if $V_{\leq d+1} \subseteq R_{\leq 1} \cdot V_{\leq d}$ for all $d \geq 0$. 
Spaces connected to 1

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Example
For $n = 2$, $V$ spanned by these monomials is connected to 1:
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Lemma
$V$ connected to 1, with basis $1 = f_1, f_2, \ldots, f_m$ of ascending degrees, write $f_i = \sum_{j=1}^{i-1} \ell_{ij} f_j$ with $\ell_{ij} \in R_{\leq 1}$. Then $V = \text{the span of the}$

$$(m-1) \times (m-1)$-$\text{subdeterminants of }$}

$$M(V) := \begin{bmatrix}
\ell_{21} & -1 \\
\ell_{31} & \ell_{32} & -1 \\
\vdots & & \ddots & \ddots \\
\ell_{m1} & \ell_{m2} & \cdots & \ell_{m,m-1} & -1
\end{bmatrix}$$
First construction

**Proposition**
Let \( V \subseteq R \) be connected to 1, of dimension \( m \), and such that \( R_{\leq 1} \cdot V \supseteq R_{\leq d} \). Then there is a uniform determinantal representation of size \( m \) for the polynomials in \( R_{\leq d} \).
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Theorem
For $n = 2$ there exist uniform $\det$ representations of size $\sim \frac{d^2}{4}$.

[Hochstenbach-Plestenjak 16]
Analysis of first construction

$V$ connected to 1 and $R_{\leq 1} \cdot V \supseteq R_{\leq d}$ imply $\dim V \geq \frac{1}{n} \binom{n + d}{n}$
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$V$ connected to 1 and $R_{≤1} \cdot V \supseteq R_{≤d}$ imply $\dim V \geq \frac{1}{n} \binom{n + d}{n}$

**Proposition**

For fixed $n$, $\exists$ uniform determinantal representation of size $\sim \frac{d^n}{n \cdot n!}$. 
Analysis of first construction

\[ V \text{ connected to } 1 \text{ and } R_{\leq 1} \cdot V \supseteq R_{\leq d} \text{ imply } \dim V \geq \frac{1}{n} \binom{n + d}{n} \]

**Proposition**

For fixed \( n \), \( \exists \) uniform determinantal representation of size \( \sim \frac{d^n}{n \cdot n!} \).

Construction uses the lattice of type \( A_{n-1} \) with generating matrix

\[
\begin{bmatrix}
2 & -1 \\
-1 & 2 & \ddots \\
& \ddots & \ddots & -1 \\
& & -1 & 2
\end{bmatrix}
\]
Analysis of first construction

$V$ connected to 1 and $R_{\leq 1} \cdot V \supseteq R_{\leq d}$ imply $\dim V \geq \frac{1}{n} \left( \binom{n+d}{n} \right)$

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-1 & 2 & \ddots & & \\
& \ddots & \ddots & -1 \\
& & -1 & 2 & \\
& & & -1 & 2
\end{pmatrix}
\]

But the exponent of $d$ is $n$ rather than $n/2$. 

(David Madore, YouTube)
Second construction: divide and conquer!

**Proposition**

Suppose $V_1, V_2 \subseteq R$ connected to 1 such that $R_{\leq 1} \cdot V_1 \cdot V_2 \supseteq R_{\leq d}$. Then there is a uniform det representation of degree-$d$ polynomials of size $-1 + \dim V_1 + \dim V_2$. 
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\begin{vmatrix}
    y & -1 \\
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\end{vmatrix}
= \sum_{i+j \leq 2} c_{ij} x^i y^j
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Suppose \( V_1, V_2 \subseteq R \) connected to 1 such that \( R_{\leq 1} \cdot V_1 \cdot V_2 \supseteq R_{\leq d} \). Then there is a uniform det representation of degree-\( d \) polynomials of size \(-1 + \dim V_1 + \dim V_2\).

**Example**

\[
\begin{bmatrix}
  x & -1 \\
  x & -1 \\
  c_{00} & c_{10} & c_{20} \\
  c_{10} & c_{11} \\
  c_{20} & & & -1 & y \\
 & -1 & y & & & -1
\end{bmatrix} = \sum_{i+j\leq 2} c_{ij} x^i y^j
\]

\( \det M(V_1) \quad M(V_2)^T \)

Can we find \( V_1, V_2 \), connected to 1, of \( \dim \sim \sqrt{\dim R_{\leq d}} \) such that \((R_1 \cdot )V_1 \cdot V_2 \supseteq R_{\leq d}\)?
A fractal

Can we find $V_1, V_2$, connected to 1, of dim growing like $\sqrt{\dim R_{\leq d}}$ such that $(R_1 \cdot) V_1 \cdot V_2 \supseteq R_{\leq d}$?
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- For $n$ even, split variables $\rightsquigarrow V_1, V_2$ of dimension $\binom{n/2 + d}{n/2}$. 


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- For $n$ even, split variables $\leadsto V_1, V_2$ of dimension $\left(\frac{n}{2} + d\right)$. 

- For odd $n$, find subsets $A_0, A_1 \subseteq (\mathbb{Z}_{\geq 0})^n$, connected to 0, of “dimension” $\frac{n}{2}$ such that $A_0 + A_1 = \mathbb{Z}_n$: 
  - start with $B_0 := \sum_{j=0}^{\infty} \{0, 1\} \cdot 2^j$; 
  - $B_1 := 2B_0$ so that $B_0 + B_1 = \mathbb{Z}_{\geq 0}$; 
  - $A_i := B_i^n$; 
  - connect to 0.
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  - start with $B_0 := \sum_{j=0}^{\infty} \{0, 1\} \cdot 2^{2j}$;
  - $B_1 := 2B_0$ so that $B_0 + B_1 = \mathbb{Z}_{\geq 0}$;
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  - connect to 0.

Take $V_i$ spanned by the monomials with exponent vectors in $A_i$. \qed
**Theorem**  [Boralevi-v Doornmalen-D-Hochstenbach-Plestenjak, 16]

For $n$ fixed, there exist $C_1, C_2$ such that a sufficiently general $p \in R_{\leq d}$ has $dc(p) \geq C_1 d^{n/2}$ and any $p \in R_{\leq d}$ has $dc(p) \leq C_2 d^{n/2}$.

Many questions remain:
- what are the best constants $C_1, C_2$?
- what about the regime where $d$ is fixed and $n$ runs?
- symmetric determinantal representations?
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Thank you!
Outlook

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Thank you!

Motivation III: hyperbolic polynomials
If $p = \det(A_0 + \sum_i x_i A_i)$ with $A_i \in \mathbb{R}^{N \times N}$ symmetric and $A_0$ positive definite, then the restriction of $p$ to any line through 0 has only real roots. For $n = 2$ the converse also holds (Helton-Vinnikov).