Catalan-many morphisms to trees, part I

Jan Draisma  
(University of Bern, TU Eindhoven)  
j.w.w. Alejandro Vargas (Bern)

Frankfurt, TGiZ, April 24, 2020
Gonality of Riemann surfaces

**Definition**

*Riemann surface*: a compact complex manifold $X$ of dimension 1.

Easiest example: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} =$
Gonality of Riemann surfaces

Definition

*Riemann surface*: a compact complex manifold $X$ of dimension 1.

Easiest example: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$

Next easiest examples: $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau) = \mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau)$
Gonality of Riemann surfaces

Definition

*Riemann surface*: a compact complex manifold $X$ of dimension 1.

Easiest example: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$

Next easiest examples: $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau) = \mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau)$

Topologically determined by their *genus*: the number of holes.
Gonality of Riemann surfaces

**Definition**

*Riemann surface*: a compact complex manifold $X$ of dimension 1.

Easiest example: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} =$

Next easiest examples: $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau) =$

Topologically determined by their *genus*: the number of holes.
Gonality of Riemann surfaces

**Definition**

*Riemann surface*: a compact complex manifold $X$ of dimension 1.

Easiest example: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = \text{genus 0}$

Next easiest examples: $\mathbb{C}/(\mathbb{Z} \mathbf{1} + \mathbb{Z} \tau) = \text{genus 1}$

Topologically determined by their *genus*: the number of holes.

**Definition**

*Gonality* of $X$: minimal degree of a holomorphic map $X \to \mathbb{P}^1$. 
Gonality of Riemann surfaces

Definition

Riemann surface: a compact complex manifold $X$ of dimension 1.

Easiest example: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = P^1$.

gonality 1  genus 0

Next easiest examples: $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau) = C/(Z1 + Z\tau) = C/(Z + Z\tau)$.

gonality 2:  genus 1

$\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left( \frac{1}{(z+m+n\tau)^2} + \frac{1}{(m+n\tau)^2} \right)$

Topologically determined by their genus: the number of holes.

Definition

Gonality of $X$: minimal degree of a holomorphic map $X \to \mathbb{P}^1$. 

Diagram:

- Gonality of $\mathbb{P}^1$.
- Gonality of $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau)$.

Diagram of a torus with a lattice and a holomorphic map to $\mathbb{P}^1$. 

Diagram of a torus with a lattice and a holomorphic map to $\mathbb{P}^1$. 

Diagram of a torus with a lattice and a holomorphic map to $\mathbb{P}^1$. 

The gonality theorem for Riemann surfaces

**Theorem** [Riemann, Kempf, Kleiman-Laksov, Griffiths-Harris, ...]

- Any \( X \) of genus \( g \) has gonality at most \( 1 + \lceil g/2 \rceil \).
- Equality for \( X \) sufficiently general.
- For \( g \) even, a sufficiently general \( X \) has precisely \( C_{g/2} \) such holomorphic maps (up to \( \text{PGL}_2 \)).

\[
C_{g/2} = 1, 2, 5, \ldots \text{ for } g = 2, 4, 6, \ldots \text{ is the } g/2\text{-th Catalan number.}
\]
The gonality theorem for Riemann surfaces

**Theorem** [Riemann, Kempf, Kleiman-Laksov, Griffiths-Harris, . . . ]

- Any $X$ of genus $g$ has gonality at most $1 + \lceil g/2 \rceil$.
- Equality for $X$ sufficiently general.
- For $g$ even, a sufficiently general $X$ has precisely $C_{g/2}$ such holomorphic maps (up to PGL$_2$).

$$C_{g/2} = 1, 2, 5, \ldots \text{ for } g = 2, 4, 6, \ldots \text{ is the } g/2\text{-th Catalan number.}$$

**Rationale for** $1 + \lfloor g/2 \rfloor$: (such a map is a divisor class of rank 1)
The gonality theorem for Riemann surfaces

**Theorem** [Riemann, Kempf, Kleiman-Laksov, Griffiths-Harris, . . .]

- Any $X$ of genus $g$ has gonality at most $1 + \lceil g/2 \rceil$.
- Equality for $X$ sufficiently general.
- For $g$ even, a sufficiently general $X$ has precisely $C_{g/2}$ such holomorphic maps (up to PGL$_2$).

$C_{g/2} = 1, 2, 5, \ldots$ for $g = 2, 4, 6, \ldots$ is the $g/2$-th Catalan number.

**Rationale for $1 + \lceil g/2 \rceil$:** (such a map is a divisor class of rank 1)

$S^d X \bigcup W = \{\text{effective deg}-d \text{ divisors of rank 1}\} \bigcap \{p = (p_1, \ldots, p_d) \in S^d X \mid \text{rk}(d_p \varphi) \leq d - 1\}$
The gonality theorem for Riemann surfaces

**Theorem** [Riemann, Kempf, Kleiman-Laksov, Griffiths-Harris, ...]

- Any $X$ of genus $g$ has gonality at most $1 + \lceil g/2 \rceil$.
- Equality for $X$ sufficiently general.
- For $g$ even, a sufficiently general $X$ has precisely $C_{g/2}$ such holomorphic maps (up to PGL$_2$).

$$C_{g/2} = 1, 2, 5, \ldots \text{ for } g = 2, 4, 6, \ldots \text{ is the } g/2\text{-th Catalan number.}$$

**Rationale for** $1 + \lfloor g/2 \rfloor$: (such a map is a divisor class of rank 1)

$S^d X$ \hspace{1cm} dimension $d$ \hspace{1cm} $\cup$ \hspace{1cm} $\varphi$ \hspace{1cm} $\cup$ \hspace{1cm} $\text{Pic}_d(X)$ \hspace{1cm} dim $g$

$W = \{\text{effective deg-}d \text{ divisors of rank 1}\}$ \hspace{1cm} $\cap$ \hspace{1cm} $\text{dim } g$

$d_p \varphi = g$

$\{p = (p_1, \ldots, p_d) \in S^d X \mid \text{rk}(d_p \varphi) \leq d - 1\}$
The gonality theorem for Riemann surfaces

**Theorem** [Riemann, Kempf, Kleiman-Laksov, Griffiths-Harris, ... ]
- Any $X$ of genus $g$ has gonality at most $1 + \lceil g/2 \rceil$.
- Equality for $X$ sufficiently general.
- For $g$ even, a sufficiently general $X$ has precisely $C_{g/2}$ such holomorphic maps (up to $\text{PGL}_2$).

$C_{g/2} = 1, 2, 5, \ldots$ for $g = 2, 4, 6, \ldots$ is the $g/2$-th Catalan number.

**Rationale for** $1 + \lfloor g/2 \rfloor$: (such a map is a divisor class of rank 1)

$S^d X$ dimension $d$ $\varphi$ $\text{Pic}_d(X)$ dim $g$

$W = \{\text{effective deg-}d \text{ divisors of rank 1} \}$ $W^1_d$ $d$

$\{p = (p_1, \ldots, p_d) \in S^d X | \text{rk}(d_p \varphi) \leq d - 1\}$

Expected dimension of $W^1_d$: $d - (g - (d - 1)) - 1$; want $\geq 0$. 

\[ \varphi \]
Metric graphs and harmonic maps

Finite, connected, 1-dim CW-complex $\Gamma$ with a suitable metric:
Finite, connected, 1-dim CW-complex $\Gamma$ with a suitable metric:

\[ g = 2 \]

\textit{genus} := first Betti number $g$
Metric graphs and harmonic maps

Finite, connected, 1-dim CW-complex $\Gamma$ with a suitable metric:

$g = \text{genus} = \text{first Betti number } g$

**Definition** [Urakawa, Baker-Norine, Caporaso, ...]
A continuous $\varphi : \Gamma \to \Sigma$ is *harmonic* if it is piecewise linear with integral slopes and $\forall v \in \Gamma$ and $e, f$ emanating from $\varphi(v)$ we have

$$\sum_{e' : \varphi(e') = e} (d\varphi)(e') = \sum_{f' : \varphi(f') = f} (d\varphi)(f')$$
Metric graphs and harmonic maps

Finite, connected, 1-dim CW-complex $\Gamma$ with a suitable metric:

$\small g = 2$

$\small genus := \text{first Betti number } g$

**Definition** [Urakawa, Baker-Norine, Caporaso, ...]

A continuous $\varphi : \Gamma \to \Sigma$ is *harmonic* if it is piecewise linear with integral slopes and $\forall v \in \Gamma$ and $e, f$ emanating from $\varphi(v)$ we have

$$
\sum_{e' : \varphi(e') = e} (d\varphi)(e') = \sum_{f' : \varphi(f') = f} (d\varphi)(f')
$$

$\small =: m_\varphi(v), \text{ local degree}$
Metric graphs and harmonic maps

Finite, connected, 1-dim CW-complex $\Gamma$ with a suitable metric:

$g = 2$

$\text{genus} := \text{first Betti number } g$

**Definition** [Urakawa, Baker-Norine, Caporaso, ... ]

A continuous $\varphi : \Gamma \to \Sigma$ is *harmonic* if it is piecewise linear with integral slopes and $\forall v \in \Gamma$ and $e, f$ emanating from $\varphi(v)$ we have

$$\sum_{e' : \varphi(e') = e} (d\varphi)(e') = \sum_{f' : \varphi(f') = f} (d\varphi)(f')$$

$=: m_\varphi(v)$, local degree

(analogue of holomorphic maps)
Metric graphs and harmonic maps

Finite, connected, 1-dim CW-complex $\Gamma$ with a suitable metric:

$g = 2$

**Definition** [Urakawa, Baker-Norine, Caporaso, ...]
A continuous $\varphi : \Gamma \rightarrow \Sigma$ is harmonic if it is piecewise linear with integral slopes and $\forall v \in \Gamma$ and $e, f$ emanating from $\varphi(v)$ we have

$$\sum_{e' : \varphi(e') = e} (d\varphi)(e') = \sum_{f' : \varphi(f') = f} (d\varphi)(f')$$

$=: m_{\varphi}(v)$, local degree

(analogue of holomorphic maps)
Finite, connected, 1-dim CW-complex $\Gamma$ with a suitable metric:

$$g = 2$$

**genus**: $= \text{first Betti number } g$

**Definition** [Urakawa, Baker-Norine, Caporaso, …]

A continuous $\varphi: \Gamma \rightarrow \Sigma$ is **harmonic** if it is piecewise linear with integral slopes and $\forall v \in \Gamma$ and $e, f$ emanating from $\varphi(v)$ we have

$$\sum_{e': \varphi(e') = e} (d\varphi)(e') = \sum_{f': \varphi(f') = f} (d\varphi)(f')$$

$=: m_{\varphi}(v)$, **local degree**

(analogue of holomorphic maps)
Tropical morphisms and gonality

**Definition**

\[ \varphi : \Gamma \to \Sigma \text{ harmonic, non-constant} \implies \deg \varphi := \sum_{v: \varphi(v) = w} m_\varphi(v) \]

(independent of \( w \in \Sigma \))
Tropical morphisms and gonality

Definition

$\varphi : \Gamma \to \Sigma$ harmonic, non-constant $\leadsto \deg \varphi := \sum_{v: \varphi(v) = w} m\varphi(v)$

(independent of $w \in \Sigma$)

Definition [Bertrand-Brugallé-Mikhalkin]

$\varphi$ is a tropical morphism if all slopes are nonzero and $\forall v \in \Gamma :$

$\text{valency}(v) - 2 \geq m\varphi(v)(\text{valency}(\varphi(v)) - 2).$  

(e.g. not allowed:)

\begin{center}
\begin{tikzpicture}
\draw[dashed] (0,0) -- (1,1);
\draw (1,1) -- (2,0);
\draw[dashed] (2,0) -- (0,0);
\node at (1,1) {2};
\node at (2,0) {2};
\node at (0,0) {2};
\end{tikzpicture}
\end{center}
Tropical morphisms and gonality

**Definition**

\( \varphi : \Gamma \to \Sigma \) harmonic, non-constant \( \implies \deg \varphi := \sum_{v : \varphi(v) = w} m_{\varphi}(v) \)

(independent of \( w \in \Sigma \))

**Definition**

[Bertrand-Brugallé-Mikhalkin]

\( \varphi \) is a *tropical morphism* if all slopes are nonzero and \( \forall v \in \Gamma : \)

\[ \text{valency}(v) - 2 \geq m_{\varphi}(v)(\text{valency}(\varphi(v)) - 2). \]

(e.g. not allowed:)

Definition

[Bertrand-Brugallé-Mikhalkin]

The (geometric) *gonality* of \( \Gamma \) is the minimal degree of any tropical morphism to a tree from a *modification* of \( \Gamma \).

**Definition**

[Mikhalkin, Caporaso, Amini, Cornelissen-Kool]

The (geometric) *gonality* of \( \Gamma \) is the minimal degree of any tropical morphism to a tree from a *modification* of \( \Gamma \).
Tropical morphisms and gonality

**Definition**

\( \varphi : \Gamma \rightarrow \Sigma \) harmonic, non-constant \( \Rightarrow \) \( \deg \varphi := \sum_{v: \varphi(v)=w} m_{\varphi}(v) \) (independent of \( w \in \Sigma \))

**Definition**

[Bertrand-Brugallé-Mikhalkin]

\( \varphi \) is a *tropical morphism* if all slopes are nonzero and \( \forall v \in \Gamma : \) valency\((v) - 2 \geq m_{\varphi}(v)(\text{valency}(\varphi(v)) - 2) \). (e.g. not allowed:)

**Definition**

[Mikhalkin, Caporaso, Amini, Cornelissen-Kool]

The (geometric) *gonality* of \( \Gamma \) is the minimal degree of any tropical morphism to a tree from a *modification* of \( \Gamma \).

*In the example, the gonality is 2:*
The gonality theorem for metric graphs

**Theorem** [Baker, Caporaso, . . . , D-Vargas]
- Any $\Gamma$ of genus $g$ has gonality at most $1 + \lceil g/2 \rceil$.
- Equality for $\Gamma$ sufficiently general.
- For $g$ even, a sufficiently general $\Gamma$ has precisely $C_{g/2}$ such tropical morphisms, when counted with multiplicities.
The gonality theorem for metric graphs

**Theorem** [Baker, Caporaso, . . . , D-Vargas]

- Any \( \Gamma \) of genus \( g \) has gonality at most \( 1 + \lceil g/2 \rceil \).
- Equality for \( \Gamma \) sufficiently general.
- For \( g \) even, a sufficiently general \( \Gamma \) has precisely \( C_{g/2} \) such tropical morphisms, when counted with multiplicities.

**Remarks**

- First item follows from the gonality theorem for Riemann surfaces via a form of Baker’s *specialisation lemma*.
- Second item follows from (first item and) work by Cools-D.
- D-Vargas is independent of these, and completely combinatorial (but \( \sim 80 \) pages).
The gonality theorem for metric graphs

**Theorem** [Baker, Caporaso, . . . , D-Vargas]

- Any $\Gamma$ of genus $g$ has gonality at most $1 + \lceil g/2 \rceil$.
- Equality for $\Gamma$ sufficiently general.
- For $g$ even, a sufficiently general $\Gamma$ has precisely $C_{g/2}$ such tropical morphisms, when counted with multiplicities.

**Remarks**

- First item follows from the gonality theorem for Riemann surfaces via a form of Baker’s specialisation lemma.
- Second item follows from (first item and) work by Cools-D.
- D-Vargas is independent of these, and completely combinatorial (but $\sim 80$ pages).

**Plan for part I:** Discuss the relation between two theorems, and part of Cools-D. Part II (Alejandro): more combinatorics.
(We follow recent work by Lionel Lang, which extends older work by Mikhalkin.)

**Lemma from hyperbolic geometry:** given $\alpha, \beta, \gamma > 0$ there exists a unique right-angled hexagon in the hyperbolic plane with side lengths $\alpha, a, \beta, b, \gamma, c$. 

![Diagram of a hyperbolic hexagon with angles $\alpha, \beta, \gamma$ and sides $a, b, c$.]
Approximating metric graphs

(We follow recent work by Lionel Lang, which extends older work by Mikhalkin.)

**Lemma from hyperbolic geometry:** given $\alpha, \beta, \gamma > 0$ there exists a unique right-angled hexagon in the hyperbolic plane with side lengths $\alpha, a, \beta, b, \gamma, c$.

Glue two copies to a **pair of pants** $P_{2\alpha, 2\beta, 2\gamma}$:
Approximating metric graphs, continued

Fix $G = (V(G), E(G))$ trivalent graph, genus $g \geq 2$, and $c \in \mathbb{R}^{E(G)}_{>0}$.
Fix $G = (V(G), E(G))$ trivalent graph, genus $g \geq 2$, and $c \in \mathbb{R}^{|E(G)|}_{>0}$.

For each $v \in V$ incident to $e_1, e_2, e_3$, take a copy $P_v$ of $P_{c(e_1), c(e_2), c(e_3)}$, and glue these to a Riemann surface $X_c$ of genus $g$: 
Limit of holomorphic maps to $\mathbb{P}^1$

Let $\ell \in \mathbb{R}^{E(G)}_{>0}$, so that $\Gamma := (G, \ell)$ is a metric graph of genus $g$. 
Let $\ell \in \mathbb{R}^{E(G)}_{>0}$, so that $\Gamma := (G, \ell)$ is a metric graph of genus $g$.

Set $c_t(e) := \frac{2\pi^2}{\ell(e) \log(t)}$ and $X_t := X_{c_t}$.

For $t \to \infty$ the Riemann surface $X_t$ degenerates into a union of $\mathbb{P}^1$s, each neighbouring three others.
Limit of holomorphic maps to $\mathbb{P}^1$

Let $\ell \in \mathbb{R}^{E(G)}_{> 0}$, so that $\Gamma := (G, \ell)$ is a metric graph of genus $g$.

Set $c_t(e) := \frac{2\pi^2}{\ell(e) \log(t)}$ and $X_t := X_{c_t}$.

For $t \to \infty$ the Riemann surface $X_t$ degenerates into a union of $\mathbb{P}^1$s, each neighbouring three others.

Let $\psi_t : X_t \to \mathbb{P}^1$; can be chosen to depend continuously on $t$.

Theorem [Mikhalkin,...,Lang]
The $\psi_t$ converge in a well-defined sense to a tropical morphism from a modification of $\Gamma$ to a tree.
Why tree and modification?

For $t \gg 0$, the images $\psi_t(C_e) =: \tilde{C}_e$ in $\mathbb{P}^1$ are disjoint.

Simplifying assumption: they are topological circles in $\mathbb{P}^1$. 
Why tree and modification?

For $t \gg 0$, the images $\psi_t(\mathcal{C}_e) =: \tilde{\mathcal{C}}_e$ in $\mathbb{P}^1$ are disjoint.

Simplifying assumption: they are topological circles in $\mathbb{P}^1$.

Create graph $T$ with $V(T) =$ connected components of $\mathbb{P}^1 \setminus \bigcup_{e \in E(G)} \tilde{\mathcal{C}}_e$; an edge if they have a common $\tilde{\mathcal{C}}_e$ in their boundary. Since $\mathbb{P}^1$ is simply connected, $T$ is a tree.
Why tree and modification?

For $t \gg 0$, the images $\psi_t(C_e) =: \tilde{C}_e$ in $\mathbb{P}^1$ are disjoint.

Simplifying assumption: they are topological circles in $\mathbb{P}^1$.

Create graph $T$ with $V(T) =$ connected components of $\mathbb{P}^1 \setminus \bigcup_{e \in E(G)} \tilde{C}_e$; an edge if they have a common $\tilde{C}_e$ in their boundary. Since $\mathbb{P}^1$ is simply connected, $T$ is a tree.

Each pre-image $\psi^{-1}(\tilde{C}_e)$ contains $C_e$ and possibly further topological circles $C_{e'}$. This yields a modification $G' = (V(G'), E(G'))$ of $G$ (with $e' \in E(G')$).
Why tree and modification?

For $t \gg 0$, the images $\psi_t(C_e) =: \tilde{C}_e$ in $\mathbb{P}^1$ are disjoint.

Simplifying assumption: they are topological circles in $\mathbb{P}^1$.

Create graph $T$ with $V(T) =$ connected components of $\mathbb{P}^1 \setminus \bigcup_{e \in E(G)} \tilde{C}_e$; an edge if they have a common $\tilde{C}_e$ in their boundary. Since $\mathbb{P}^1$ is simply connected, $T$ is a tree.

Each pre-image $\psi^{-1}(\tilde{C}_e)$ contains $C_e$ and possibly further topological circles $C_{e'}$. This yields a modification $G' = (V(G'), E(G'))$ of $G$ (with $e' \in E(G')$).

In the limit, we find a tropical morphism from a modification $\Gamma'$ of $\Gamma$ with combinatorial type $G'$ to a metric tree with combinatorial type $T$. 
Why balancing and Riemann-Hurwitz?

Pick a vertex $v \in V(G')$; this corresponds to a connected component $U$ of $X_t \setminus \bigcup_{e \in E(G')} C_e$.

Assume no loops at $v$. Then $\overline{U}$ is $\mathbb{P}^1$ minus $k$ discs corresponding to the edges incident to $v$; Euler characteristic: $2 - k$. 
Why balancing and Riemann-Hurwitz?

Pick a vertex $v \in V(G')$; this corresponds to a connected component $U$ of $X_t \setminus \bigcup_{e \in E(G')} C_e$.

Assume no loops at $v$. Then $\overline{U}$ is $\mathbb{P}^1$ minus $k$ discs corresponding to the edges incident to $v$; Euler characteristic: $2 - k$.

Let $W := \psi_t(U)$; so $\overline{W}$ is $\mathbb{P}^1$ minus $l$ discs; Euler char. $2 - l$. 
Why balancing and Riemann-Hurwitz?

Pick a vertex $v \in V(G')$; this corresponds to a connected component $U$ of $X_t \setminus \bigcup_{e \in E(G')} C_e$.

Assume no loops at $v$. Then $\overline{U}$ is $\mathbb{P}^1$ minus $k$ discs corresponding to the edges incident to $v$; Euler characteristic: $2 - k$.

Let $W := \psi_t(U)$; so $\overline{W}$ is $\mathbb{P}^1$ minus $l$ discs; Euler char. $2 - l$.

For each $\tilde{C}_e, e \in E(G)$ in the boundary of $\overline{W}$ and $C_{e'}, e' \in E(G')$ in the boundary of $\overline{U}$ mapping onto $\tilde{C}_e$, let $m_{e'}$ be the degree of that map $C_{e'} \to \tilde{C}_e$. These are the slopes in our tropical morphism; $m_\varphi(v)$ is the degree of the branched cover $\psi_t|_U : \overline{U} \to \overline{W}$. 
Why balancing and Riemann-Hurwitz?

Pick a vertex \( v \in V(G') \); this corresponds to a connected component \( U \) of \( X_t \setminus \bigcup_{e \in E(G')} C_e \).

Assume no loops at \( v \). Then \( \bar{U} \) is \( \mathbb{P}^1 \) minus \( k \) discs corresponding to the edges incident to \( v \); Euler characteristic: \( 2 - k \).

Let \( W := \psi_t(U) \); so \( \bar{W} \) is \( \mathbb{P}^1 \) minus \( l \) discs; Euler char. \( 2 - l \).

For each \( \tilde{C}_e, e \in E(G) \) in the boundary of \( \bar{W} \) and \( C_{e'}, e' \in E(G') \) in the boundary of \( \bar{U} \) mapping onto \( \tilde{C}_e \), let \( m_{e'} \) be the degree of that map \( C_{e'} \to \tilde{C}_e \). These are the slopes in our tropical morphism; \( m_\phi(v) \) is the degree of the branched cover \( \psi_t|_U : \bar{U} \to \bar{W} \).

**R-H formula:** \( 2 - k = m_\phi(v)(2 - l) - \sum_{p \in U}(e_p - 1) \); so \( (k - 2) \geq m_\phi(v) \cdot (l - 2) \)
Moduli space of genus-\(g\) metric graphs

Let \(g \geq 2\).

For each ordinary genus-\(g\) graph \(G = (V, E)\) set \(C_G := (\mathbb{R}_{>0})^E\).

For any isomorphism \(G \rightarrow H\) glue \(C_G\) to \(C_H\).

If contracting \(e\) in \(G\) yields a genus-\(g\) graph \(H\), glue \(C_H\) to \(C_G\) as the boundary with \(e\)-th coordinate 0.

Identify modifications (ignore dangling trees).

\(\rightsquigarrow\) yields the moduli space \(M_g\) of genus-\(g\) metric graphs.
Moduli space of genus-g metric graphs

- Let $g \geq 2$.
- For each ordinary genus-$g$ graph $G = (V, E)$ set $C_G := (\mathbb{R}_{>0})^E$.
- For any isomorphism $G \to H$ glue $C_G$ to $C_H$.
- If contracting $e$ in $G$ yields a genus-$g$ graph $H$, glue $C_H$ to $C_G$ as the boundary with $e$-th coordinate 0.
- Identify modifications (ignore dangling trees).

$\Rightarrow$ yields the moduli space $M_g$ of genus-$g$ metric graphs.

If $G$ trivalent, then $\dim C_G = |E| = 3g - 3 \Rightarrow \dim M_g = 3g - 3$.

Caporaso: $M_g$ connected in codimension 1.
Theorem
For $d, g \geq 2$ the gonality-$d$ locus in $M_g$ is locally closed of dim $\min\{3g - 3, 2g + 2d - 5\}$ \textit{(perhaps not pure-dim)}. In particular, the locus where the gonality is $\geq 1 + \lceil g/2 \rceil$ is dense and open.
Older combinatorial results by Cools-D

**Theorem**
For $d, g \geq 2$ the gonality-$d$ locus in $M_g$ is locally closed of dim $\min\{3g - 3, 2g + 2d - 5\}$ (perhaps not pure-dim). In particular, the locus where the gonality is $\geq 1 + \lceil g/2 \rceil$ is dense and open.

**Theorem**
For each trivalent combinatorial type $G$, the preimage in $C_G$ of the gonality-$1 + \lceil g/2 \rceil$ locus contains an open cone.
Theorem
For $d, g \geq 2$ the gonality-$d$ locus in $M_g$ is locally closed of dim $\min\{3g - 3, 2g + 2d - 5\}$ (perhaps not pure-dim). In particular, the locus where the gonality is $\geq 1 + \lceil g/2 \rceil$ is dense and open.

Theorem
For each trivalent combinatorial type $G$, the preimage in $C_G$ of the gonality-$1 + \lceil g/2 \rceil$ locus contains an open cone.

Remarks
• Dimension matches the classical count for curves.
• Via approximation, the first Theorem implies that a general genus-$g$ Riemann surface has gonality (at least) $1 + \lceil g/2 \rceil$ — no need for a specific graph to prove this. (Observed by Mikhalkin in 2011.)
Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $1 + \lceil g/2 \rceil =: d$ (2nd Theorem).

- Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
- Case 1: $G$ is cactus (any two simple cycles intersect in at most one point) $\Rightarrow$ any point $\Gamma$ in $C_G$ has gonality $\leq d$. 
Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality \(1 + \lceil g/2 \rceil =: d\) (2nd Theorem).

- Let \(G = (V, E)\) be a trivalent graph with \(|E| - |V| + 1 = g\).
- Case 1: \(G\) is cactus (any two simple cycles intersect in at most one point) \(\leadsto\) any point \(\Gamma\) in \(C_G\) has gonality \(\leq d\).

- Case 2: \(G\) has a trivalent vertex \(v\) such that \(G' = G - v\) is connected, of genus \(g - 2\). By induction, \(C_{G'}\) contains an open cone, of dimension \(3g - 9\), where the gonality is \(d - 1\). We glue in a tripod, as follows:
Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $1 + \lceil g/2 \rceil =: d$ (2nd Theorem).

- Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
- Case 1: $G$ is cactus (any two simple cycles intersect in at most one point) $\Rightarrow$ any point $\Gamma$ in $C_G$ has gonality $\leq d$.

- Case 2: $G$ has a trivalent vertex $v$ such that $G' = G - v$ is connected, of genus $g - 2$. By induction, $C_{G'}$ contains an open cone, of dimension $3g - 9$, where the gonality is $d - 1$. We glue in a tripod, as follows:
Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $1 + \lceil g/2 \rceil =: d$ (2nd Theorem).

- Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
- Case 1: $G$ is cactus (any two simple cycles intersect in at most one point) $\Rightarrow$ any point $\Gamma$ in $C_G$ has gonality $\leq d$.

- Case 2: $G$ has a trivalent vertex $v$ such that $G' = G - v$ is connected, of genus $g - 2$. By induction, $C_{G'}$ contains an open cone, of dimension $3g - 9$, where the gonality is $d - 1$. We glue in a tripod, as follows:
Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $1 + \lceil g/2 \rceil =: d$ (2nd Theorem).

- Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
- Case 1: $G$ is cactus (any two simple cycles intersect in at most one point) $\implies \text{any point } \Gamma \text{ in } C_G \text{ has gonality } \leq d$.

- Case 2: $G$ has a trivalent vertex $v$ such that $G' = G - v$ is connected, of genus $g - 2$. By induction, $C_{G'}$ contains an open cone, of dimension $3g - 9$, where the gonality is $d - 1$. We glue in a tripod, as follows:
Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $1 + \lceil g/2 \rceil =: d$ (2nd Theorem).

- Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
- Case 1: $G$ is cactus (any two simple cycles intersect in at most one point) $\implies$ any point $\Gamma$ in $C_G$ has gonality $\leq d$.

- Case 2: $G$ has a trivalent vertex $v$ such that $G' = G - v$ is connected, of genus $g - 2$. By induction, $C_{G'}$ contains an open cone, of dimension $3g - 9$, where the gonality is $d - 1$. We glue in a tripod, as follows:
Constructing metric graphs with prescribed gonality

Concentrate on the case of gonality $1 + \lceil g/2 \rceil =: d$ (2nd Theorem).

- Let $G = (V, E)$ be a trivalent graph with $|E| - |V| + 1 = g$.
- Case 1: $G$ is cactus (any two simple cycles intersect in at most one point) $\Rightarrow$ any point $\Gamma$ in $C_G$ has gonality $\leq d$.

- Case 2: $G$ has a trivalent vertex $v$ such that $G' = G - v$ is connected, of genus $g - 2$. By induction, $C_{G'}$ contains an open cone, of dimension $3g - 9$, where the gonality is $d - 1$. We glue in a tripod, as follows:

Parameter count:
3 for the gray dots
3 for the orange edges
$3g - 9 + 3 + 3 = 3g - 3$  □
• The morphism on the right moves in a family of dimension $3g - 3$: the map from edge lengths of the tree to edge lengths in $G$ is a linear bijection.
• The morphism on the right moves in a family of dimension $3g - 3$: the map from edge lengths of the tree to edge lengths in $G$ is a linear bijection.

• The morphism on the right imposes certain lower bounds on the red edges: if they are all sufficiently long, the construction works.
• The morphism on the right moves in a family of dimension $3g - 3$: the map from edge lengths of the tree to edge lengths in $G$ is a linear bijection.

• The morphism on the right imposes certain lower bounds on the red edges: if they are all sufficiently long, the construction works.

• What happens when we shrink one of the new leaf edges? Can we change the morphism so that the red edge can shrink further?
• The morphism on the right moves in a family of dimension $3g - 3$: the map from edge lengths of the tree to edge lengths in $G$ is a linear bijection.

• The morphism on the right imposes certain lower bounds on the red edges: if they are all sufficiently long, the construction works.

• What happens when we shrink one of the new leaf edges? Can we change the morphism so that the red edge can shrink further?

Answer: YES, see Alejandro’s talk next!