Tropical Brill-Noether theory

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The B(aker)-N(orin) game on graphs

Requirements
finite, connected, undirected graph $\Gamma$
d $\geq 0$ chips
natural number $r$

Rules
B puts $d$ chips on $\Gamma$
N demands $\geq r_v \geq 0$ chips at $v$ with $\sum_v r_v = r$
B wins iff he can fire to meet N’s demand
Brill-Noether theorems for graphs

\[ g := e(\Gamma) - v(\Gamma) + 1 \text{ genus of } \Gamma \]
\[ \rho := g - (r + 1)(g - d + r) \]

**Conjecture (Matthew Baker)**
1. \( \rho \geq 0 \Rightarrow \text{B has a winning starting position.} \)
2. \( \rho < 0 \Rightarrow \text{B may not have one, depending on } \Gamma. \)
   \((\forall g \exists \Gamma \forall d, r : \rho < 0 \Rightarrow \text{Brill loses.})\)

**Theorem (Baker / Caporaso)**
1. is true.
   \((uses \text{ sophisticated algebraic geometry})\)

**Theorem (Cools-D-Payne-Robeva)**
2. is true.
   \((implies \text{ sophisticated algebraic geometry})\)
Chip dragging on graphs

Simultaneously moving all chips along edges, with zero net movement around every cycle.

Lemma
1. Chip dragging is realisable by chip firing.
2. W.l.o.g. B *drags* instead of *firing*.

Example 1: $\Gamma$ a tree
\[ \rho = g - (r+1)(g-d+r) = -(r+1)(-d+r) \]
B wins $\iff \rho \geq 0 \iff d \geq r$

Example 2: a hyperelliptic graph
\[ d = 2, \ r = 1 \]
Who wins?
The B(rill)-N(oether) game on Riemann surface

Requirements
compact Riemann surface \( X \)
d chips
natural number \( r \)

Rules
B puts \( d \) chips on \( X \)
N demands \( \geq r_x \geq 0 \) chips at \( x \) with \( \sum_x r_x = r \)
B wins iff he can drag to meet N’s demand
Chip dragging on Riemann surfaces

Simultaneously moving chips \( c \) along paths \( \gamma_c : [0, 1] \to X \), such that
\[
\sum_c \langle \omega|_{\gamma(t)}, \gamma'_c(t) \rangle = 0
\]
for all holomorphic 1-forms \( \omega \) on \( X \).

Lemma
\[
D = \sum_c [\gamma_c(0)] \text{ initial position}
\]
\[
E = \sum_c [\gamma_c(1)] \text{ final position}
\]
\[
\iff E - D \text{ is divisor of meromorphic function on } X
\]
drag-equivalence = linear equivalence

Example: torus
only one holomorphic 1-form: \( dz \)
condition: \( \sum_c \gamma'_c(t) = 0 \)
when does B win?
Dimension count

$\omega_1, \ldots, \omega_g$ basis of holomorphic 1-forms

$x = (x_1, \ldots, x_d) \in X \times \cdots \times X$

$v_i \neq 0$ tangent vector at $x_i$

$\leadsto$ matrix $A_x = (\langle \omega_i, v_j \rangle)_{ij} \in \mathbb{C}^{g \times d}$

$(c_1v_1, \ldots, c_dv_d)$ infinitesimal dragging direction $\Rightarrow A(c_1, \ldots, c_d)^T = 0$

$x$ winning for $B \Rightarrow$

dragging $x$ fills $\geq r$-dimensional variety

where $\ker A$ is $\geq r$-dimensional

$\#$ conditions on $g \times d$-matrix to have

$\geq r$-dimensional kernel: $r(g - d + r)$

for $B$ to have a winning position, “need”

$d - r(g - d + r) \geq r$

$\Leftrightarrow \rho = g - (r + 1)(g - d + r) \geq 0$
Brill-Noether theorems for Riemann surfaces

Theorem (Meis 1960, Kempf 1971, Kleiman-Laksov 1972)
\( \rho \geq 0 \Rightarrow \text{B has a winning position.} \)

Theorem (Griffiths-Harris 1980)
1. \( \rho < 0 \Rightarrow \text{B may lose, depending on } X. \)
\( \forall g \exists X \forall d, r : \rho < 0 \Rightarrow \text{B loses.} \)

2. \( \rho \geq 0 \) and \( X \) general
\( \Rightarrow \rho = \dim \{ \text{winning positions modulo dragging} \} \)

3. \( \rho = 0 \) and \( X \) general
\( \Rightarrow \# = \# \text{ standard tableaux of shape } \)
\( (r + 1) \times (g - d + r) \) with entries 1, 2, \ldots, g
Specialisation

Algebro-geometric (Baker, Caporaso)
dual graph of special fibre
applies to arbitrary fields
integral starting positions for $B$

Complex-analytic (Mikhalkin-Zharkov)
conceptually simpler?
rational starting positions for $B$
\{${X_t}$\}_{t \neq 0} family of Riemman surfaces
$\rightsquigarrow \Gamma$ for $t \to 0$ (“tropical limit”)
holomorphic 1-forms on $X_t \rightsquigarrow$ “1-forms” on $\Gamma$
chip dragging on $X_t \rightsquigarrow$ chip dragging on $\Gamma$

Theorem
$D_t$ winning for $B$ on $X_t$ and $D_t \to D$ on $\Gamma$ for $t \to 0$
$\Rightarrow D$ winning on $\Gamma$. 
Consequences of Specialisation

Meis/Kempf/Kleiman-Laksov
(\( \rho \geq 0 \) implies B wins on Riemann surfaces)
⇒ same statement for \( \Gamma \).
No combinatorial proof is known!

Cools-D-Payne-Robeva
(\( \rho < 0 \) ⇒ B loses for suitable \( \Gamma \))
⇒ same for Riemann surfaces (Griffiths-Harris 1 and 2, and probably 3).

Technical difficulties:
1. Find family \( \{X_t\}_t \) with
dual graph \( \Gamma \) (algebro-geometric) or
degenerating to \( \Gamma \) (complex-analytic);
2. show that \( t \mapsto D_t \) (winning position on \( X_t \))
can be chosen such that \( D_t \) “converges”.
Example (Cools-D-Payne-Robeva)

\[ g = 4, d = 3, r = 1 \]
\[ g - d + r = 2 \]
\[ r + 1 = 2, \rho = 0 \]

\[
\begin{array}{cc}
1 & 3 \\
2 & 4 \\
\end{array}
\sim 1, 2, 3, 2, 1
\]

\[
\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}
\sim 1, 2, 1, 2, 1
\]
A larger example

\[ g = 7, d = 7, r = 2 \]
\[ \implies g - d + r = 2, r + 1 = 3, \rho = 1 \]

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 6 & 7 \\
\end{array}
\implies (21, 31, 32, 42, 31, 31, 32, 21) \text{ lingering lattice path}
\]

Theorem (Cools-D-Payne-Robeva)
B’s starting position \( \implies \) lingering lattice path in \( \mathbb{Z}^r \);
B wins iff path stays in chamber \( \{(x_1, \ldots, x_r) \mid x_1 > x_2 > \ldots > x_r > 0\} \).

\( \implies \rho \geq 0 \iff \) B wins
Castryck and Cools’s gonality conjecture

\[ r = 1 \]
\[ f \in \mathbb{C}[x, y] \text{ general with Newton polytope } \Delta \]
\[ X := \{ f = 0 \} \text{ Riemann surface} \]

**Conjecture**

minimal \( d \) for which \( B \) wins on \( X := \{ f = 0 \} \)

(=minimal degree of a meromorphic map to \( \mathbb{P}^1 \))
equals \( d = \text{lattice width of } \Delta \)
(with two exceptions)
Purely combinatorial?

Theorem (van der Pol)
\[ \rho \geq 0 \text{ and } \Gamma \text{ a cactus graph} \]
\[ \Rightarrow B \text{ has winning positions with all chips at vertices.} \]

Future goal:
Understand Kleiman-Laksov for (metric) graphs.
Baker’s Specialisation Lemma

\(\mathcal{X}\) curve family over \(\mathbb{C}[[t]]\)
(proper, flat, regular scheme)
generic fibre \(\mathcal{X}_{\mathbb{C}((t))}\) smooth curve \(X\)
special fibre \(\mathcal{X}_{\mathbb{C}} = X_1 \cup \ldots \cup X_s\)
\(X_i\) smooth, intersections simple nodes
\(\leadsto\) dual graph \(\Gamma\) on \(\{u_1, \ldots, u_s\}\)
(metric with edge lengths \(1\))
\(\leadsto\) map \(X(\mathbb{C}((t)))) \rightarrow \{u_1, \ldots, u_s\}\)
well-behaved with respect to finite extensions \(\mathbb{C}((t^{1/n}))/\mathbb{C}((t))\)
\(\leadsto\) specialisation map \(\tau : X(\mathbb{C}\{\{t\}\}) \rightarrow \Gamma_\mathbb{Q}\)

**Theorem**
Brill wins with starting positing \(D\) on \(X(\mathbb{C}\{\{t\}\})\)
\(\Rightarrow\) Baker wins with starting position \(\tau(D)\) on \(\Gamma_\mathbb{Q}\)
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