Symmetries and $\infty$-dim limits of algebro-statistical models

Jan Draisma

TU Eindhoven
Part I: some infinite-dimensional commutative algebra
What is an infinite-dimensional variety?

*inductive limits of finite-dimensional varieties, projective limits, spectra of infinite-dimensional rings, etc.*
What is an infinite-dimensional variety?

Inductive limits of finite-dimensional varieties, projective limits, spectra of infinite-dimensional rings, etc.

This talk

$V$: countable-dimensional space over $\mathbb{C}$ (or $\mathbb{R}$) of coordinates $V^*$: dual space, topological space with Zariski topology

Closed subsets $X \subseteq V^*$ are called infinite-dimensional varieties.

Example

$V = \langle x_{ij} \mid i, j \in \mathbb{N} \rangle$, $X \subseteq V^*$ defined by equations $x_{ij}x_{kl} - x_{il}x_{kj}$
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Sequence model

If $V_1 \subseteq V_2 \subseteq \ldots$ finite-dimensional with $V = \bigcup_i V_i$, then $V^* = \lim_{\leftarrow} V_i^*$ with $V_1^* \hookleftarrow V_2^* \hookleftarrow \ldots$

(both as set and as topological space)
Noetherianity modulo group actions

Assume a group $G$ acts by linear transformations on $V \overset{\sim}{\twoheadrightarrow} G$ acts on $SV$ by algebra auto and on $V^*$ by homeo.
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$\sim$ G acts on $SV$ by algebra auto and on $V^*$ by homeo.

For $X \subseteq V^*$ a closed set, $\mathbb{C}[X] := SV/I(X)$, where $I(X) \subseteq SV$ is the ideal of polynomials vanishing on $X$. 
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**Definition**

$\mathbb{C}[X]$ is $G$-Noetherian if every chain $I_1 \subseteq I_2 \subseteq \ldots$ of $G$-stable ideals stabilises

$(\Leftrightarrow$ each $G$-stable ideal is $G$-finitely generated.$)$
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(Non-)Examples of G-Noetherianity

Finite-by-infinite matrices
Fix $k \in \mathbb{N}$;
$\text{Sym}(\mathbb{N})$ acts on $V = \langle x_{ij} \mid i \in [k], j \in \mathbb{N} \rangle$
by $\pi(x_{ij}) = x_{i\pi(j)}$. 
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**Theorem** [Cohen 87, Hillar-Sullivant 09]

$\mathbb{C}[x_{ij} \mid i \in [k], j \in \mathbb{N}] = \mathbb{C}[V^*]$ is Sym($\mathbb{N}$)-Noetherian.
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**Infinite-by-infinite matrices**

Sym$(\mathbb{N})$ acts by $\pi(x_{ij}) = x_{\pi(i),\pi(j)}$
$\leadsto \mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]$ is *not* Sym$(\mathbb{N})$-Noetherian;
e.g. the Sym$(\mathbb{N})$-stable ideal generated by $x_{12}x_{21}$, $x_{12}x_{23}x_{31}$, $x_{12}x_{23}x_{34}x_{41}$, ...  
is not Sym$(\mathbb{N})$-finitely generated.

*(neither Sym$(\mathbb{N}) \times$ Sym$(\mathbb{N})$-Noetherian)*
(Non-)Examples, continued

**Theorem (Matrices of bounded rank)**

$\mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]/(\text{all } (k + 1) \times (k + 1)-\text{subdeterminants})$ is Sym($\mathbb{N}$)-Noetherian.

*(uses $2k \times \mathbb{N}$-matrices and the FFT, SFT for $\text{GL}_k$)*
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Larger groups

\( \text{GL}_\mathbb{N} := \{ \text{invertible } \mathbb{N} \times \mathbb{N} \text{-matrices } g \text{ with almost all } g_{ii} = 1 \text{ and almost all } g_{ij} = 0(i \neq j) \} \).
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Theorem (symmetric matrices) [Snowden-Sam 2012]
\[ \mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}, x_{ij} = x_{ji}] \text{ is } GL_\mathbb{N}-\text{Noetherian via } g \circ x = gxg^T. \]
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**Theorem** \[ \text{[D-Eggermont 2014]} \]
\[ (\mathbb{C}^{\mathbb{N} \times \mathbb{N}})^p \text{ is } \text{GL}_\mathbb{N} \times \text{GL}_\mathbb{N}-\text{Noetherian for each } p, \text{ via} \]
\[ (g, h) \circ (x, \ldots, z) := (gxh^{-1}, \ldots, gzh^{-1}). \]
Example: second hypersimplex
\[ P_n := \{ v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n \} \]
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**Markov basis** \( M_n \)  

\[ v_{ij} = v_{ji} \text{ and } v_{ij} + v_{kl} = v_{il} + v_{kj} \text{ for } i, j, k, l \text{ distinct} \]

\( \leadsto \text{ if } \sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij} \text{ with } c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}, \)

then the expressions are connected by such

*moves* without creating negative coefficients

[De Loera-Sturmfels-Thomas 1995]
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Theorem \hfill [D-Eggermont-Krone-Leykin 2013]

For any family \( (P_n \subseteq \mathbb{Z}^F \times \mathbb{Z}^{k\times n}) \), \( F \) finite, if \( P_n = \text{Sym}(n)P_{n_0} \) for \( n \geq n_0 \), then \( \exists n_1: \) for \( n \geq n_1 \) has a Markov basis \( M_n \) with \( M_n = \text{Sym}(n)M_{n_0} \).
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Explicit results for width \( n_0 = 2 \): [Kahle-Krone-Leykin 2014]
Part II: Applications to algebro-statistical models
Constructing limits

Setting

$V_1^*, V_2^*, \ldots$ fin-dim spaces; $X_i \subseteq V_i^*$ subvariety

$G_i$ group acting linearly on $V_i^*$ preserving $X_i$

$G_i \subseteq G_{i+1}$ & maps $\pi: V_{i+1}^* \rightarrow V_i^*$ and $\iota: V_i^* \rightarrow V_{i+1}^*$ both $G_i$-equivariant, mapping $X_{i+1}$ into $X_i$ and v.v. & $\pi \circ \iota = \text{id}$
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Definition

Sequence $(X_i \subseteq V_i^*)_i$ stabilises if for $n \gg 0$:

$p \in V_n^*$ lies in $X_n$ iff $\forall g \in G_n \pi(gp) \in X_{n-1}$. 
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$V_\infty := \lim_\leftarrow V_n$; $X_\infty := \lim_\leftarrow X_n$; $G_\infty := \cup_n G_n$

Lemma Stabilisation is “equivalent” to: $X_\infty \subseteq V_\infty^*$ is defined by finitely many $G_\infty$-orbits of equations.
I: The independent set theorem

Fixed row and column sums

$A, B \in \mathbb{Z}_{\geq 0}^{m \times n}$ with $a_{i+} = b_{i+}$ and $a_{+j} = b_{+j}$

$\Rightarrow \exists A = A_0, A_1, \ldots, A_k = B \in \mathbb{Z}_{\geq 0}^{m \times n}$ with

$A_l - A_{l-1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \leadsto \text{moves "independent" of } m, n.$
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Theorem [Diaconis-Sturmfels 1998]

basis of Markov moves = generating set of toric ideal

(e.g. ker$[y_{ij} \mapsto x_{i}z_{j}]$ generated by $\{y_{ij}y_{i'j'} - y_{ij}y_{i'j}\}$)
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\((\text{e.g. ker}[y_{ij} \mapsto x_i z_j] \text{ generated by } \{y_{ij}y_{i'j'} - y_{ij}y_{i'j'}\})\)

Conjecture [Hoşten-Sullivant 2007]
Similar stabilisation conjecture for Markov basis for sampling higher-dimensional contingency tables.
Hierarchichal models

$F$ family of subsets of $[m]\n
y(i_1, \ldots, i_m)$ and $x(S, (i_s)_{s \in S})$ for $S \in F$ variables

$I := \ker[y(i_1, \ldots, i_m) \mapsto \prod_{A \in S} x(S, (i_s)_{s \in S})]\n
Example

$m = 4, F = \{124, 13, 23\}\n
variables y(abcd), x(abd), z(ac), u(bc)\n
I = \ker[y(abcd) \mapsto x(abd)z(ac)u(bc)]\n
\begin{tikzpicture}[scale=0.5]
    \draw (0,0) -- (4,0) -- (2,2) -- (0,4) -- cycle;
    \fill[green!50!black] (0,0) -- (4,0) -- (2,2) -- (0,4) -- cycle;
    \draw[fill=black] (0,0) circle (0.1);
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    \node at (0,0) {1};
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\end{tikzpicture}
The independent set theorem, continued

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Theorem

[ Hillar-Sullivant 2012 ]

If $T \subseteq [m]$ independent set ($|T \cap S| \leq 1$ for $S \in F$);

$i_t, t \in T$ run through $\mathbb{N}$ and $i_t, t \notin T$ through $[r_t]$

$\leadsto I$ generated by finitely many Inc($\mathbb{N}$)-orbits

(now this also follows from D-Eggermont-Krone-Leykin)
II: Cloning sinks in a Gaussian Bayesian model

$G$: directed acyclic graph on $[n]$

$X_i, i \in [n]$: jointly Gaussian

$X_j = \sum_{i \in \text{pa}(j)} \lambda_{ij} X_i + a_j \epsilon_j$ where the $\epsilon_j \sim N(0, 1)$ independent

$\Sigma = (I - \Lambda)^{-T} \text{diag}(a_1^2, \ldots, a_n^2)(I - \Lambda)^{-1}$
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\(H \subseteq [m] \text{ hidden} \sim \Sigma_{[n]-H} \text{ principal submatrix model: Zariski closure of } \{\Sigma_{[n]-H}|\Lambda, a\}\)
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$H \subseteq [m]$ hidden $\leadsto \Sigma_{[n]-H}$ principal submatrix

model: Zariski closure of $\{\Sigma_{[n]-H}|\Lambda, a\}$

Cloning sinks

$\begin{array}{c}
\text{Diagram with arrows pointing to } j_1, j_2
\end{array}$
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Cloning sinks

Model stabilises under cloning sinks (via permuting clones).
Stabilisation for parameterised graphical models

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**I: stabilises under increasing $r_j$ for $j$ in independent set**

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I: stabilises under increasing $r_j$ for $j$ in independent set

II: stabilises under cloning sinks

III: stabilises under cloning sinks?? Yes for trees.

|  | Gaussian mean 0 |
|  | $\{ \sum = (I - \Lambda)^{-T} D (I - \Lambda)^{-1} \}$ |
|  | $\Lambda_{ij} = 0 \text{ if } i \not\leftrightarrow j$ |
|  | $M(G) \subseteq \mathbb{C}^{([n]-H) \times ([n]-H)}$ |

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<td>$\forall i \in \mathbb{R}<em>{\text{pa}(j)} : \sum</em>{i_j} \theta_{i_j</td>
</tr>
<tr>
<td>I: stabilises under increasing $r_j$ for $j$ in independent set</td>
<td>II: stabilises under cloning sinks?? Yes for trees.</td>
</tr>
<tr>
<td>$M(G) := { \Sigma = K^{-1} }$</td>
<td>$\Sigma = (I - \Lambda)^{-T} D (I - \Lambda)^{-1}$</td>
</tr>
<tr>
<td>$K_{ij} = 0$ if $ij \notin E(G)$</td>
<td>$\Lambda_{ij} = 0$ if $i \nrightarrow j$</td>
</tr>
<tr>
<td>$M(G) \subseteq \mathbb{C}^{n \times n}$</td>
<td>$M(G) \subseteq \mathbb{C}^{([n]-H) \times ([n]-H)}$</td>
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</tbody>
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Definition
\[ C \in \mathbb{R}_{\geq 0}^{m \times n} \leadsto \text{rk}_{\geq 0} C := \min \{ r \mid \exists (A, B) \in \mathbb{R}_{\geq 0}^{m \times r} \times \mathbb{R}_{\geq 0}^{r \times n} : C = AB \} \]
IV: Nonnegative matrix rank

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Very ill-behaved

For all \( n > 3 \) there are \( n \times n \)-matrices of nonnegative rank > 3 with all proper submatrices of nonnegative rank 3.

[Moitra 2012]
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(positive cone over) mixture of \( r \) copies of independence

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Observation/Theorem \[\text{[Kubjas, Robeva, Sturmfels 2013]}\]
EM-algorithm for \( \mathcal{M}_{r}^{m \times n} \) often converges to boundary!
Explicit, quantifier-free expression for \( r = 2 \).
Nonnegative matrix rank, continued

Algebraic boundary
\[ \partial M_{m \times n}^r : \text{Zariski closure} \subseteq \mathbb{C}^{m \times n} \]
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Nonnegative matrix rank, continued

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**Theorem** [Kubjas-Robeva-Sturmfels 2013]

Apart from coordinate hyperplanes, for \( m, n \geq 4 \), \( \partial M_3^{m \times n} \) has 2 \( \text{Sym}(m) \times \text{Sym}(n) \)-orbits of irreducible components, parameterised by the following and its transpose:

\[
\begin{bmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0 \\
* & * & *
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & * & * & * \\
* & * & 0 & * \\
* & * & * & 0 & * \\
* & * & * & * & *
\end{bmatrix}
\]
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* & * & * & *
\end{bmatrix}
\]

**Conjecture**
This component has a GB of \(4 \times 4\) minors plus \(\binom{m}{3}\) sextics.
Nonnegative matrix rank, continued

**Algebraic boundary**

$\partial M_{m \times n}^r$: Zariski closure $\subseteq \mathbb{C}^{m \times n}$ hypersurface in the variety of rank-$r$ matrices

**Theorem** [Kubjas-Robeva-Sturmfels 2013]

Apart from coordinate hyperplanes, for $m, n \geq 4$, $\partial M_{3 \times n}^3$ has 2 Sym($m$) $\times$ Sym($n$)-orbits of irreducible components, parameterised by the following and its transpose:

\[
\begin{bmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0 \\
* & * & * \\
\end{bmatrix}
\quad \begin{bmatrix}
0 & 0 & * & * & * \\
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Now a **Theorem** due to Eggermont-Horobeț-Kubjas.

**But what about higher nonnegative rank??**
Conclusions and questions

- Many algebro-statistical models fit into families with a meaningful limit.

- There is an ever growing body of commutative algebra for dealing with these limits up to symmetry.

- Do discrete Bayesian models stabilise under cloning sinks?

- Do undirected Gaussian graphical models exhibit any kind of stabilisation?

- If you have other families of models where you expect stabilisation, come talk to me!
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Děkuji!