Tropical aspects of algebraic matroids

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Two advertisements

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Algebraic matroids

$K$ algebraically closed, $X \subseteq K^E$ irreducible

$\leadsto$ algebraic matroid $M(X)$ on $E$:
$I \subseteq E$ independent $:\iff X \to K^I$ dominant

Example (Fano and non-Fano): $K = \overline{\mathbb{F}_2}$,
$X = \text{im}[K^3 \to K^7, (x, y, z) \mapsto (x, y, z, y + z, x + z, x + y, x + y + z)]$

{algebraic matroids} is closed under deletion and contraction

BIG OPEN QUESTIONS:
• is algebraic realisability decidable?
• is the class closed under duality?
• how many algebraic matroids are there?
Application: matrix completion

**General observation:** \( I \) independent \( \iff \) \( \text{Trop}(X) \rightarrow \mathbb{R}^I \) surjective.

\( X_{m,n,r} = \{ \text{matrices of rank} \leq r \} \subseteq \mathbb{K}^{m \times n} \)

\( I \subseteq [m] \times [n] \) independent \( \iff \) a general partial \( I \)-matrix /\( K \) can be completed to a rank-\( \leq r \)-matrix.

\( r = 1 : \)

**Open:** can independence in \( M(X_{m,n,r}) \) be tested in poly time?

**Theorem (Daniel Bernstein):** For \( r = 2 \), \( I \) is independent iff \( I \) has an acyclic orientation without alternating cycles.

Proof uses \( \text{Trop(Gr}_{2,m+n}) \subseteq \text{Trop} \{ \text{skew-symmetric matrices} \} \).
Lemma (Ingleton)
If char\(K = 0\), then \(\{\text{matroids algebraic}/K\} = \{\text{matroids linear}/K\}\).

Proof: For general \(v \in X\), \(M(T_vX) = M(X)\).

\(\leadsto\) answers in char 0: yes, yes, 0 percent (Nelson, 2017).

Does not work in char \(p > 0\):

Example
\(X = \{(t, t^p) \mid t \in K\}\), \(M(X)\) has bases \(\{1\}, \{2\}\) 
but \(T_vX = \langle(1, 0)\rangle\) so \(M(T_vX)\) only has basis \(\{1\}\).

Way out (Lindström): replace \(X\) by \(\{(x_1^p, x_2) \mid x \in X\} = Y = \{(t, t) \mid t \in K\}\) and \(T_vY = \langle(1, 1)\rangle\) with \(M(T_vY) = M(Y) = M(X)\).
An algebraic but nonlinear matroid

only linear in char 2

\[\Rightarrow M(X)\text{ is a nonlinear but algebraic matroid}\]

\[\Rightarrow\text{ cannot find a } Y \subseteq K^{10} \text{ with } v \in Y \text{ such that } M(T_vY) = M(X)\]

Way out: Frobenius flocks!
Frobenius flocks

\[ F : a \mapsto a^p \] the Frobenius automorphism
\[ \mathbb{Z}^E \] acts on \( K^E \) via \( \alpha v := (F^{-\alpha_i}v_i)_{i \in E} \) by Zariski-homeomorphisms

For \( X \subseteq K^E \) and \( \alpha \in \mathbb{Z}^E \) have \( M(X) = M(\alpha X) \).

**Theorem (Bollen–Draisma–Pendavingh)**
For general \( v \in X \), the map \( V : \mathbb{Z}^E \to \text{Gr}(d, K^E), V(\alpha) = T_{\alpha v} \alpha X \) has the following properties:

(FF1) \( V_{\alpha} \cap e_i^\perp = \text{diag}(1, \ldots, 1, 0, 1, \ldots, 1)V_{\alpha + e_i} \)

(FF2) \( V_{\alpha + 1} = 1V_\alpha \)

and moreover \( \text{Bases}(M(X)) = \bigcup_\alpha \text{Bases}(M(V(\alpha))) \).

**Definition**
A map \( V \) satisfying (FF1), (FF2) is called a *Frobenius flock*. 
Example 1

\((\text{FF1})\) \(V_\alpha \cap e_i^\perp = \text{diag}(1, \ldots, 1, 0, 1, \ldots, 1)V_{\alpha + e_i}\) \(\text{(FF2)}\) \(V_{\alpha + 1} = 1V_\alpha\)

\((X =) V_0 = \langle (1, a) \rangle, \ a \neq 0\)

\(V_0 \cap e_1^\perp = \{(0, 0)\} = \text{diag}(0, 1)V_{e_1} \leadsto V_{e_1} = \langle (1, 0) \rangle\)

FF2 yields:
Example 2

\[ X = \{(x, y, x + y, x + y^{(p^g)}) \mid (x, y) \in K^2\} \subseteq K^4, \ g > 1, \ M(X) = U_{2,4} \]

\[ T_0X = \text{row space of } \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \text{ so } 1, 4 \text{ parallel in } M(T_0X). \]

\[ (-e_2 - e_3)X = \{(x, y, x^p + y, x + y^{(p^{g-1})}) \mid (x, y) \in K^2\} \]

\[ T_0(-e_2 - e_3)X = \text{row space of } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}; \text{ also } 2, 3 \text{ parallel.} \]

\[ (-ge_2 - ge_3)X = \{(x, y, x^{(p^g)} + y, x + y) \mid (x, y) \in K^2\} \]

\[ T_0(-ge_2 - ge_3)X = \text{row space of } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}; \text{ 1, 4 indep.} \]
Example 2, continued

Cells where $M(T_{\alpha x}(\alpha X))$ is constant:

These cells are *alcoved polytopes*: max-plus and min-plus closed.
Matroid flocks from valuations and vice versa

Definition (Dress-Wenzel)
A matroid valuation is a map \( \nu : \{d\text{-sets in } E\} \to \mathbb{R} \cup \{\infty\} \) such that \( \nu(B) \neq \infty \) for some \( B \) and \( \forall B, B', i \in B \setminus B' \ \exists j \in B' \setminus B : \nu(B) + \nu(B') \geq \nu(B - i + j) + \nu(B' + i - j) \).

(\( \nu \) then lies in the Dressian and defines a tropical linear space)

Observations
\( \nu \mapsto \) two matroids: \( M^{\nu} := \{B \mid \nu(B) < \infty\} \) and \{\( B \mid \nu(B) \) minimal\}; and \( \nu'(B) := \nu(B) - \alpha \cdot e_B \) is a valuation for each \( \alpha \in \mathbb{R}^E \).

Theorem (Bollen-Draisma-Pendavingh)
Given a \( \mathbb{Z} \cup \{\infty\}\)-valued \( \nu \), set \( M^{\nu}_\alpha := \{B \mid \nu(B) - \alpha \cdot e_B \) minimal\} for each \( \alpha \in \mathbb{Z}^E \). This satisfies matroid analogues of FF1, FF2. Conversely, each such matroid flock arises in this manner.
So to a $d$-dimensional algebraic variety $X \subseteq K^E$ in char $p$ we associate the *Lindstrom valuation* $\nu^X : \{d\text{-subsets of } E\} \to \mathbb{Z} \cup \{\infty\}$. Cartwright found a direct construction of $\nu^X$. 
Monomial parameterisations

\( \varphi : (K^*)^d \to (K^*)^n \) monomial map, \( \varphi(t) = (t^{Ae_1}, t^{Ae_2}, \ldots, t^{Ae_n}) \), where \( A \in \mathbb{Z}^{d \times n} \). Set \( X := \text{im} \varphi \).

**Theorem:**
\( \nu^X \) sends \( B \subseteq [n] \), \( |B| = d \) to the \( p \)-adic valuation of \( \det A[B] \).

Generalises? \( G \) a 1-dimensional algebraic group defined over \( \mathbb{F}_p \).
\( E := \text{End}(G) \) has \( F \in E \).
\( A \in E^{d \times n} \leadsto \) a \( d \)-dimensional subgroup \( X \subseteq G^n \)

**Theorem (I think):**
\( \nu_X(B) = \text{number of factors } F \text{ in the Smith normal form of } A. \)
Rigidity

**Definition (Dress-Wenzel)**
A matroid $M$ is *rigid* if every valuation $\nu$ with $M' = M$ is of the form $M \rightarrow \mathbb{R}$, $B \mapsto \alpha \cdot e_B$ for some $\alpha \in \mathbb{R}^E$.

**Theorem**
A rigid matroid is algebraically representable over an algebraically closed field $K$ of positive characteristic if and only if it is linearly representable over $K$.

**Proof**
If $X$ is an algebraic representation, then the Lindström valuation $\nu^X : M(X) \rightarrow \mathbb{Z}$ sends $B \mapsto \alpha \cdot e_B$ for some $\alpha \in \mathbb{Z}^E$. Then $M'_\alpha = M'$. Now $M(X) = M(T_{\alpha x} \alpha X)$ for $x \in X$ general. $\square$

*Applies to projective planes over finite fields!*
Properties of Frobenius flocks

Frobenius flocks . . .

• have deletion/contraction
• are *almost* preserved under duality (replace $F$ by $F^{-1}$)
• allow for circuit hyperplane relaxations
• so Vamos is Frobenius flock realisable (and many more!):  


Thank you!