Orthogonally decomposable tensors as semisimple algebras

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With:
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Every $A \in \mathbb{C}^{m \times n}$ can be written as $A = \sum_{i=1}^{k} u_i v_i^T$ with nonzero and pairwise perpendicular $u_1, \ldots, u_k \in \mathbb{C}^m$ and similar $v_1, \ldots, v_k \in \mathbb{C}^n$.

*(Perpendicular w.r.t. the standard Hermitian forms (\langle . | . \rangle) on $\mathbb{C}^m, \mathbb{C}^n$—but ordinary transposition.)*
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If $m = n$ and $A$ is skew, then $k = 2\ell$ and one can take $v_i = u_{i+\ell}$ for $i \leq \ell$ and $v_i = -u_{i-\ell}$ for $i > \ell$; then $A = \sum_{i=1}^{\ell} (u_i v_i^T - v_i u_i^T)$. 
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Question. Which tensors admit orthogonal decompositions?
Orthogonally and unitarily decomposable tensors

Fix fin.-dim. inner product spaces $V, V_1, \ldots, V_d$ over $K \in \{\mathbb{R}, \mathbb{C}\}$. 

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**Definition.** A tensor $T \in V_1 \otimes \cdots \otimes V_d$ is **odeco/udeco** if it can be written as $T = \sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{id}$ where for each $j = 1, \ldots, d$ the vectors $v_{1j}, \ldots, v_{kj}$ are nonzero and pairwise perpendicular.
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**Definition.** A symmetric tensor $T \in \text{Sym}^d(V) \subseteq V^\otimes d$ is *symmetrically odeco/udeco* if it can be written as $T = \sum_{i=1}^{k} \pm v_{i}^\otimes d$ for nonzero, pairwise perpendicular $v_i$. 
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**Example.** With $V = \mathbb{R}^{2}$ and $d = 3$ the tensor

$$T = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$$

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**Example.** With $V = \mathbb{R}^2$ and $d = 3$ the tensor $T = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$ is symmetrically odeco: $T = (e_0 + e_1)^\otimes 3 / 2 + (e_0 - e_1)^\otimes 3 / 2$
Definition. An alternating tensor $T \in \text{Alt}^d V \subseteq V \otimes^d$ is \textit{alternatingly odeco/udeco} if $T = \sum_{i=1}^{k} v_{i1} \wedge \cdots \wedge v_{id}$ for $k \cdot d$ nonzero, pairwise perpendicular vectors $v_{11}, \ldots, v_{kd}$.

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Main theorem. For $d \geq 3$ odeco/udeco tensors form a real-algebraic variety defined by polynomials of the following degrees:

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**Proof** (symmetrically odeco case). For $V = \mathbb{R}^n$ consider

$$([v_1|\cdots|v_n], \lambda) \mapsto \sum_{i=1}^n \lambda_i v_i^\otimes d$$

$$O_n \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}(\text{Sym}^d V)$$

The lhs is compact, so the image is closed, and its pre-image in \text{Sym}^d (V) \setminus \{0\} is the set of nonzero sym odeco tensors. \qed
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**Proposition.** For $d \geq 3$ the orthogonal decomposition is unique.

**Proof** (ordinary case). Contracting $T = \sum_{i=1}^k v_{i1} \otimes \cdots \otimes v_{id}$ with a general tensor in $V_3 \otimes \cdots \otimes V_d$ yields a two-tensor $A$ with distinct nonzero singular values.  \qed
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(This yields an algorithm for orthogonal decomposition—Kolda.)
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**Proposition.** An odeco symmetric tensor is symmetrically odeco.  
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**Conjecture (Robeva).** This characterises ordinary odeco tensors.
Main theorem for symmetrically odec0 three-tensors

Via the isomorphism $V^\otimes 3 \cong V^* \otimes V^* \otimes V$, a $T \in \text{Sym}^3(V) \subseteq V^\otimes 3$ gives rise to a bilinear map $V \times V \to V$, $(u, v) \mapsto u \cdot v = uv$. Note: $uv = vu$ since $(12)T = T$; and $(uv|w) = (uw|v)$ since $(23)T = T$. 
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$\Leftarrow$ may assume $(V, \cdot)$ is simple. Pick $x$ such that $M_x : y \mapsto xy$ is nonzero. Then ker $M_x$ is an ideal, so 0. Define $y \ast z := M_x^{-1}(yz)$. $\Rightarrow (V, \ast)$ is simple, comm, ass, with 1 and compatible $(.|.)$, so $\cong \mathbb{R}$.  

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The proof is very similar, except now $T \in U \otimes V \otimes W$ gives rise to a commutative algebra structure on $U \oplus V \oplus W$ with $U \cdot V \subseteq W$, $U \cdot U = \{0\}$, etc., and we are interested only in homogeneous ideals.
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*Partial associativity* means that $(xy)z = x(yz)$ whenever $x, y, z$ are homogeneous and $x, z$ belong to the same space $(U, V, W)$. 
Again, $T \in \text{Alt}^3(V)$ gives a bilinear multiplication $(x, y) \mapsto xy$. Now we have $xy = -yx$ and $(xy|z) = -(xz|y)$.
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**Proposition.** $T$ is alternatingly odeco iff $(V, \cdot)$ satisfies the Jacobi identity and furthermore has the property that for each $x, y, z \in V$ the map $C := M_x M_{yz} + M_y M_{zx} + M_z M_{xy}$ centralises all $M_u$. 
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**Proof.** \( \Rightarrow: \) \( V \) decomposes as an orthogonal direct sum of copies of \((\mathbb{R}^3, \times)\), for which the expression above is the Casimir element.
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**Proof.** $\Rightarrow$: $V$ decomposes as an orthogonal direct sum of copies of $(\mathbb{R}^3, \times)$, for which the expression above is the Casimir element.

$\Leftarrow$: $(V, \cdot)$ is then a compact Lie algebra. Their classification implies that the only simple one for which $C$ is central, is $(\mathbb{R}^3, \times)$. $\square$
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We have a similar characterisation for ordinary three-tensors.
What about tensors of order > 3?

**Ordinary case.** For \( d \geq 4 \), a tensor in \( V_1 \otimes \cdots \otimes V_d \) is odeco/udeco iff its flattening into \((\bigotimes_{i \in I_1} V_i) \otimes \cdots \otimes (\bigotimes_{i \in I_e} V_i)\) is for each partition \( I_1, \ldots, I_e \) of \( \{1, \ldots, d\} \) with at least one \( |I_j| > 1 \).
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This proves the main theorem, except ...
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<tr>
<td><strong>alternating</strong></td>
<td>2 (Jacobi), 4 (Casimir)</td>
<td>3,4??</td>
</tr>
</tbody>
</table>

There is a 280-dimensional space of cubic equations for udeco tensors in $\text{Alt}^3 \mathbb{C}^6$, one of which looks like:

$$
t_{1,4,5}t_{2,3,4}\bar{t}_{1,3,5} - t_{1,3,4}t_{2,4,5}\bar{t}_{1,3,5} + t_{1,2,4}t_{3,4,5}\bar{t}_{1,3,5} + t_{1,4,6}t_{2,3,4}\bar{t}_{1,3,6} - t_{1,3,4}t_{2,4,6}\bar{t}_{1,3,6} + t_{1,2,4}t_{3,4,6}\bar{t}_{1,3,6} - t_{1,4,6}t_{2,4,5}\bar{t}_{1,5,6} + t_{1,4,5}t_{2,4,6}\bar{t}_{1,5,6} - t_{1,2,4}t_{4,5,6}\bar{t}_{1,5,6} + t_{2,4,6}t_{3,4,5}\bar{t}_{3,5,6} - t_{2,4,5}t_{3,4,6}\bar{t}_{3,5,6} + t_{2,3,4}t_{4,5,6}\bar{t}_{3,5,6}$$

...but the algebra has *no* polynomial identities of degree 3 :-(