POLARISATION IN INVARIANT THEORY

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INVARIANT THEORY

Set-up:
- \( V \) a finite-dimensional vector space over a field \( K \)
- \( G \) a group acting by linear maps on \( V \)
- \( K[V] \) the ring of polynomials on \( V \)
- \( G \) acts on \( K[V] \) by \( g f := f \circ g^{-1} \)
- \( K[V]^G \) the algebra of (polynomial) \( G \)-invariants on \( V \)

Example 0.1. \( V := \mathbb{K}^n \), \( G := \{ g \in \text{GL}_n \mid g^T g = I \} \); then \( K[V]^G = K[x_1, \ldots, x_n]^G = K[q] \), where \( q = x_1^2 + \ldots + x_n^2 \).

Invariants help to distinguish \( G \)-orbits in \( V \):
- If \( x, y \in V \) and \( f \in K[V]^G \) satisfies \( f(x) \neq f(y) \), then \( Gx \) and \( Gy \) are distinct orbits.
- For finite groups, distinct orbits are separated by invariant polynomials, but this is not true for general groups.

Two problems in invariant theory

Generators. Given \( G \) and its action on \( V \), determine generators of \( K[V]^G \).

Remark 0.2. • \( K[V]^G \) may not be a finitely generated algebra; this is the negative answer to Hilbert’s 14th problem (Nagata 1959).
• If the image of \( G \) in \( \text{GL}(V) \) is “reductive”, then \( K[V]^G \) is finitely generated (Hilbert, 1890), and one can give good upper bounds on the degrees of generators (Derksen, 2001).

Separating invariants. Given \( G \) and its action on \( V \), determine a finite subset of \( K[V]^G \) with the same “separating power” as the full algebra \( K[V]^G \). Such a set is called a finite separating system of invariants.

Lemma 0.3. A finite separating system of invariants always exists.

Proof. Consider the equivalence relation on \( V \) given by

\[ \{(v, w) \mid f(v) = f(w) \text{ for all } f \in K[V]^G \}. \]

This is the zero set of the ideal \( I \) generated by the polynomials \( f(v) - f(w) \), where \( f \) runs over all elements of \( K[V]^G \). By the fact that \( K[V \times V] \) is Noetherian (i.e., every ascending chain of ideals stabilises), \( I \) is already generated by a finite number of the \( f(v) - f(w) \). The corresponding \( f \) form a finite separating system. \( \square \)
Note that a finite set of invariants has the same separating power as the algebra it generates. So we may also interpret the lemma as: there always exists a finitely generated algebra of invariants that has the same separating power as the full algebra $K[V]^G$.

**Polarisation**

Now suppose that $V = M^p$, where

- $M$ is a finite-dimensional vector space on which $G$ acts linearly, and
- $G$ acts diagonally on $M^p$.

In general one cannot construct all invariants on $M^p$ from invariants on $M^q$, as follows. Suppose that $f \in K[M^q]^G$ and let $\pi$ be any linear map $K^p \rightarrow K^q$ by

$$
\pi(m_1, \ldots, m_p) = (\sum_{j=1}^p \pi_{1,j} m_j, \ldots, \sum_{j=1}^p \pi_{q,j} m_j).
$$

Clearly, this map is $G$-equivariant: $\pi(gv) = g\pi(v)$ for $v \in M^p$. Hence $f \circ \pi$ is an invariant on $M^p$, which we will call a polarisation of $f$ to $M^p$.

**Definition 0.5.** Let $A$ be a subalgebra of $K[M^q]$. The subalgebra of $K[M^p]$ generated by all polarisations of elements of $A$ to $M^p$ (i.e., by all functions of the form $f \circ \pi$ where $f \in A$ and $\pi : K^p \rightarrow K^q$) is called the polarisation of $A$ to $M^p$.

**Example 0.6.** Recall Example 0.1. Then the polynomial $\beta$ on $(K^n)^2$ given by

$$
\beta(x, y) := q(x + y) - q(x) - q(y)
$$

is a polarisation of $q$ to 2 copies of $K^n$: the bilinear form associated to $q$.

**Applications of polarisation**

**Generating invariants.** The following theorem is due to Weyl.

**Theorem 0.7** (Weyl, 1939). Suppose that char $K = 0$. Then $K[M^p]^G$ is the polarisation of $K[M^q]^G$ to $M^p$ for all $p \geq \dim M$.

In other words, one needs “only” know the invariants of $G$ on $\dim M$ copies of $M$ to construct the invariants of $G$ on more copies. Compare this to example 0.4.

The theorem is not true in positive characteristic, not even if $G$ is finite and its order is not a multiple of char $K$.

**Example 0.8.** (Kemper, Wehlau). Suppose that $K$ has characteristic 3 and contains a primitive 4-th root of unity. Let $G$ be the subgroup of $GL_1$ generated by $\omega$, acting on $M = K$ by multiplication. Then $K[M]^G = K[x]^G = K[x^4].$ Now $x^2y^2$ is an invariant on $M^2$, but in $(ax + by)^4$ the monomial $x^2y^2$ does not occur; this easily implies that $x^2y^2$ does not lie in the polarisation of $K[x^4]$ to 2 copies.
Remark 0.9. Friedrich Knop has proved a generalisation of Weyl’s theorem to positive characteristic, which ensures that the invariants on \( M^p \) up to a certain degree can be obtained from invariants on \( M^{\dim M} \) by polarisation. Combining this with a bound on degrees of invariants of finite groups, one finds a generalisation of Weyl’s theorem to finite \( G \) and \( \text{char} \ K \) not dividing \( |G| \).

Separating invariants. In contrast, we have

Theorem 0.10 (Kemper, Wehlau, Draisma, 2005). If \( A \) is a separating subalgebra of \( K[M^{\dim M}]^G \), then the polarisation of \( A \) to \( M^p \) with \( p \geq \dim M \) is separating.

This theorem is a consequence of the following lemma (by taking “having the same value under all invariants” as equivalence relations).

Lemma 0.11. Suppose \( p, q \geq \dim M \) and let \( \sim \) and \( \equiv \) be equivalence relations on \( M^p \) and \( M^q \), respectively, such that 
\[
\forall v, w \in M^p, \pi : K^p \to K^q, \quad v \sim w \Rightarrow \pi v \equiv \pi w,
\]
and 
\[
\forall v, w \in M^q, \pi : K^q \to K^p, \quad v \equiv w \Rightarrow \pi v \sim \pi w.
\]
Now suppose that \( v, w \in M^p \) are such that \( \pi v \equiv \pi w \) for all \( \pi : K^p \to K^q \). Then \( v \sim w \).

Proof. View \( M^p \) and \( M^q \) as \( K^p \otimes M \) and \( K^q \otimes M \). Choose linearly independent subspaces \( A, B, C \) of \( K^p \) with

1. \( v \in (A + B) \otimes M \) (write \( v = v_A + v_B \) accordingly),
2. \( w \in (B + C) \otimes M \) (write \( w = w_B + w_C \) accordingly) and
3. \( A + B \) and \( B + C \) have dimension at most \( q \).

Now let \( \pi : K^p \to K^q \) and \( \sigma : K^q \to K^p \) be such that \( \sigma \pi \) is the identity on \( A + B \) and zero on \( C \). Then we find 
\[
v = \sigma \pi v \sim \sigma \pi w = w_B.
\]
Similarly, using a second pair \( (\sigma, \pi) \), we find \( w \sim v_B \). But now let \( \pi : K^p \to K^q \) and \( \sigma : K^q \to K^p \) be a third pair such that \( \sigma \pi \) is the identity on \( B \) and zero on \( A + C \). Then we find 
\[
v_B = \sigma \pi v \sim \sigma \pi w = w_B,
\]
and we are done. \( \square \)

The null-cone. The null-cone \( N(M^p) \) in \( M^p \) is the set of elements of \( V \) that cannot be separated from 0 by invariants.

Example 0.12. Suppose that \( \text{SL}_n \times \text{SL}_n \) acts by left-and-right multiplication on \( M = M_n \).

- The null-cone \( N(M) \) consists of the singular matrices: these are the ones that cannot be distinguished from 0 by the det.
- The null-cone \( N(M^p) \) for \( p > 1 \) has precisely \( n \) irreducible components, namely: 
  \[
  C_k := \{(A_1, \ldots, A_p) \in M^p \mid \exists k \text{-dimensional } U : \dim \sum_i A_i U < k \}.
  \]

Theorem 0.13 (Bürgin, Draisma, 2005). The function \( p \mapsto \text{"the number of irreducible components of } N(M^p) \)" is ascending and stabilises at some \( p \leq \dim M \).
Remark 0.14. For reductive groups in characteristic zero, this was first observed by Kraft and Wallach (2004).

Proof. The stabilising part is the hardest. Set \( q := \dim M \). Consider the map

\[ \Psi : \text{Hom}(K^q, K^p) \times M^q \to M^p, \quad (\pi, v) := \pi v. \]

Verify:

1. \( \Psi \) maps \( \text{Hom}(K^q, K^p) \times N(M^q) \) into \( N(M^p) \),
2. \( \Psi \) maps \( \text{Hom}(K^q, K^p) \times N(M^q) \) onto \( N(M^p) \) (here we need \( q = \dim M \)), and
3. the number of irreducible components of \( \text{Hom}(K^q, K^p) \times N(M^q) \) equals the number of irreducible components of \( N(M^q) \).

\( \square \)