Higher Secants of Sato’s Grassmannian

Jan Draisma
TU Eindhoven

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Grassmannians: functoriality and duality

$V$ a fin-dim vector space over an infinite field $K$
$\mapsto \text{Gr}_p(V) := \{v_1 \wedge \cdots \wedge v_p | v_i \in V\} \subseteq \wedge^p V$
cone over Grassmannian
(rank-one alternating tensors)
Grassmannians: functoriality and duality

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Two properties:

1. if $\varphi : V \to W$ linear

$\sim \wedge^p \varphi : \wedge^p V \to \wedge^p W$

maps $\mathbf{Gr}_p(V) \to \mathbf{Gr}_p(W)$
Grassmannians: functoriality and duality

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Define $Gr_p(V) := \{v_1 \wedge \cdots \wedge v_p \mid v_i \in V\} \subseteq \wedge^p V$

cone over Grassmannian

(rank-one alternating tensors)

Two properties:

1. If $\varphi : V \to W$ is linear,$\quad \implies \wedge^p \varphi : \wedge^p V \to \wedge^p W$

maps $Gr_p(V) \to Gr_p(W)$

2. If $\dim V =: n + p$ with $n, p \geq 0$,$\quad \implies$ natural map $\wedge^p V \to (\wedge^n V)^* \to \wedge^n (V^*)$

maps $Gr_p(V) \to Gr_n(V^*)$
Plücker varieties

**Definition**

**Rules** $X_0, X_1, X_2, \ldots$ with

\[ X_p : \{ \text{vector spaces } V \} \to \{ \text{varieties in } \bigwedge^p V \} \]
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**Definition**

Rules $X_0, X_1, X_2, \ldots$ with

$$X_p : \{\text{vector spaces } V\} \rightarrow \{\text{varieties in } \wedge^p V\}$$

form a *Plücker variety* if, for dim $V = n + p$,

1. $\varphi : V \rightarrow W \rightsquigarrow \wedge^p \varphi$ maps $X_p(V) \rightarrow X_p(W)$
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Constructions
$X, Y$ Plücker varieties $\rightsquigarrow$ so are
$X + Y$ (join), $\tau X$ (tangential),
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**Constructions**
$X, Y$ Plücker varieties $\leadsto$ so are

$X + Y$ (join), $\tau X$ (tangential),
$X \cup Y, X \cap Y$

*skew analogue of Snowden’s $\Delta$-varieties*
Definition
A Plücker variety \{X_p\}_p is \textit{bounded} if \(X_2(V) \neq \bigwedge^2 V\) for \(\text{dim } V\) sufficiently large.
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Main Theorem
Any bounded Plücker variety is defined set-theoretically in bounded degree, by finitely many equations \textit{up to symmetry}. 
Results, with Eggermont

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Theorem
For any fixed bounded Plücker variety there exists a polynomial-time membership test.
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Theorems apply, in particular, to \( k\text{Gr} = \{ \text{alternating tensors of alternating rank } \leq k \} \)
Noetherianity up to symmetry

**Definition**

A ring $R$ with a group $G$ acting on it is $G$-Noetherian if every $G$-stable ideal of $R$ is generated by finitely many $G$-orbits.
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**Theorem (Cohen 1987, Hillar-Sullivant 2009)**
$R = K[x_{ij} | i = 1, \ldots, k; \ j \in \mathbb{N}]$ is $G$-Noetherian for $G = \text{Sym}(\mathbb{N})$ with $\pi x_{ij} = x_{i\pi(j)}$. 


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*Lots of applications: algebraic statistics, multilinear algebra, . . . but not needed today.*
Definition
A topological space $X$ with a group $G$ acting on it is \textit{$G$-Noetherian} if every chain $X \supseteq X_1 \supseteq X_2 \supseteq \ldots$ of $G$-stable closed subsets stabilises.
Topological Noetherianity up to symmetry

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Remark
If \( R \) is a \( G \)-Noetherian \( K \)-algebra, then \( \text{Hom}_{K-\text{alg}}(R, K) \) \( G \)-Noetherian topological space with Zariski topology.
(Converse not true!)
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**Remark**
If $R$ is a $G$-Noetherian $K$-algebra, then $\text{Hom}_{K-\text{alg}}(R, K)$ is a $G$-Noetherian topological space with Zariski topology. *(Converse not true!)*

**Constructions**
- $G$-stable subsets, and $G$-equivariant images, and finite unions of $G$-Noetherian spaces are $G$-Noetherian.
- If $G$ acts on $X$ and $Y \subseteq X$ is $H$-Noetherian for some subgroup $H \subseteq G$, then $GY$ is $G$-Noetherian.
Tuples of infinite-by-infinite matrices

**Theorem**

Set $\text{GL}_\mathbb{N} := \bigcup_{n \in \mathbb{N}} \text{GL}_n(K)$ and $M_\mathbb{N} := K^{\mathbb{N} \times \mathbb{N}}$. For any $N \in \mathbb{N}$, $M^N_\mathbb{N}$ is $\text{GL}_\mathbb{N} \times \text{GL}_\mathbb{N}$-Noetherian with the Zariski topology.
Theorem
Set \( GL_N := \bigcup_{n \in \mathbb{N}} GL_n(K) \) and \( M_N := K^{N \times N} \). For any \( N \in \mathbb{N} \), \( M_N^N \) is \( GL_N \times GL_N \)-Noetherian with the Zariski topology.

Key notion
\( A_1, \ldots, A_N \) matrices of same sizes (perhaps infinite)
\( \rightsquigarrow \text{rk}(A_1, \ldots, A_N) := \min \left\{ \text{rk} \left( \sum c_i A_i \right) \mid (c_1 : \ldots : c_N) \in \mathbb{P}^{N-1} \right\} \)
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**Proof idea**
$M_\mathbb{N}^N \supseteq X_1 \supseteq X_2 \supseteq \ldots$ closed, $\text{GL}_\mathbb{N} \times \text{GL}_\mathbb{N}$-stable
\[ \rightsquigarrow \text{either sup}_{A \in X_n} \text{rk}(A) < \infty \text{ for } n \gg 0, \text{ or } = \infty \text{ for all } n \]
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**Proof idea**
$M_N^N \supseteq X_1 \supseteq X_2 \supseteq \ldots$ closed, $\text{GL}_\mathbb{N} \times \text{GL}_\mathbb{N}$-stable
$\leadsto$ either $\sup_{A \in X_n} \text{rk}(A) < \infty$ for $n \gg 0$, or $= \infty$ for all $n$

1st case: $X_n$ lies in image of $M_N^{N-1} \times$ “small stuff”, induction
Tuples of infinite-by-infinite matrices

**Theorem**
Set \( \text{GL}_N := \bigcup_{n \in \mathbb{N}} \text{GL}_n(K) \) and \( M_N := K^{N \times N} \). For any \( N \in \mathbb{N} \), \( M_N \) is \( \text{GL}_N \times \text{GL}_N \)-Noetherian with the Zariski topology.

**Key notion**
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\( \mapsto \text{rk}(A_1, \ldots, A_N) := \min \left\{ \text{rk} \left( \sum c_i A_i \right) \mid (c_1: \ldots: c_N) \in \mathbb{P}^{N-1} \right\} \)

**Proof idea**
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1st case: \( X_n \) lies in image of \( M_{N-1} \times \text{“small stuff”}, \) induction

2nd case: \( X_n = M_N \) for all \( n \)
Back to Plücker varieties

Definition
Rules \( X_0, X_1, X_2, \ldots \) with \( X_p : V \mapsto \) a variety in \( \bigwedge^p V \) form a Plücker variety if

1. \( \varphi : V \to W \mapsto \bigwedge^p \varphi \) maps \( X_p(V) \to X_p(W) \)

2. \( \bigwedge^p V \to \bigwedge^n(V^*) \) maps \( X_p(V) \to X_n(V^*) \)

\( \{X_p\}_p \) is bounded if \( \exists V : X_2(V) \neq \bigwedge^2 V \)
Definition
Rules $X_0, X_1, X_2, \ldots$ with $X_p : V \mapsto$ a variety in $\bigwedge^p V$ form a Plücker variety if
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$\{X_p\}_p$ is bounded if $\exists V : X_2(V) \neq \bigwedge^2 V$

Main Theorem
Any bounded Plücker variety is defined set-theoretically in bounded degree, by finitely many equations up to symmetry.

(Not scheme-theoretically. Perhaps with Landsberg-Ottaviani’s skew flattenings?)
Back to Plücker varieties

**Definition**
Rules $X_0, X_1, X_2, \ldots$ with $X_p : V \mapsto$ a variety in $\Lambda^p V$ form a Plücker variety if

1. $\varphi : V \to W \leadsto \Lambda^p \varphi$ maps $X_p(V) \to X_p(W)$
2. $\Lambda^p V \to \Lambda^n(V^*)$ maps $X_p(V) \to X_n(V^*)$

$\{X_p\}_p$ is bounded if $\exists V : X_2(V) \neq \Lambda^2 V$

**Main Theorem**
Any bounded Plücker variety is defined set-theoretically in bounded degree, by finitely many equations *up to symmetry*.

(Not *scheme-theoretically*. Perhaps with Landsberg-Ottaviani’s *skew flattenings*?)

**Approach:** organise all $X_p(V)$ into one infinite-dimensional space.
The infinite wedge

\[ V_\infty := \langle \ldots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \ldots \rangle_K \]
\[ V_{n,p} := \langle x_{-n}, \ldots, x_{-1}, x_1, \ldots, x_p \rangle \subseteq V_\infty \]
The infinite wedge

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Diagram

\[
\begin{array}{ccc}
\wedge^0 V_{00} & \wedge^1 V_{01} & \wedge^2 V_{02} \\
\downarrow & \downarrow & \downarrow \\
\wedge^0 V_{10} & \wedge^1 V_{11} & \wedge^2 V_{12} \\
\downarrow & \downarrow & \downarrow \\
\wedge^p V_{n+1,p} & \wedge^{p+1} V_{n,p+1} \\
\downarrow & \downarrow \\
\end{array}
\]
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Diagram

\[ \bigwedge^0 V_{00} \rightarrow \bigwedge^1 V_{01} \rightarrow \bigwedge^2 V_{02} \rightarrow \]
\[ \downarrow \downarrow \downarrow \]
\[ \bigwedge^0 V_{10} \rightarrow \bigwedge^1 V_{11} \rightarrow \bigwedge^2 V_{12} \rightarrow \]
\[ \bigwedge^p V_{np} \rightarrow \bigwedge^{p+1} V_{n,p+1} \]
\[ t \mapsto t \wedge x_{p+1} \]
\[ \bigwedge^p V_{n+1,p} \rightarrow \]
The infinite wedge

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Definition

\[ \wedge^{\infty / 2} V_\infty := \lim_{n \to \infty} \wedge^p V_{n,p} \text{ the infinite wedge (charge-0 part);} \]

basis \( \{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots \}_I, \ I = \{i_1 < i_2 < \ldots\}, \ i_k = k \text{ for } k \gg 0 \)
The infinite wedge

\[ V_\infty := \langle \ldots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \ldots \rangle_K \]
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Definition

\[ \wedge^{\infty/2} V_\infty := \lim_{\rightarrow} \wedge^p V_{n,p} \text{ the infinite wedge (charge-0 part);} \]
\[ \text{basis } \{ x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots \} I, \ I = \{ i_1 < i_2 < \ldots \}, i_k = k \text{ for } k \gg 0 \]

On \[ \wedge^{\infty/2} V_\infty \text{ acts } \text{GL}_\infty := \bigcup_{n,p} \text{GL}(V_{n,p}). \]
Recall
\( \bigwedge^{\infty/2} V_{\infty} \) has basis \( \{ x_{I} := x_{i_{1}} \wedge x_{i_{2}} \wedge \cdots \}_{I} \), where
\( I = \{ i_{1} < i_{2} < \ldots \} \subseteq (-\mathbb{N}) \cup (+\mathbb{N}) \) with \( i_{k} = k \) for \( k \gg 0 \).
Recall
\( \bigwedge_{\infty/2} V_{\infty} \) has basis \( \{ x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots \}_I \), where
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Bijection with Young diagrams
\( x_I \) with \( I = \{-3, -2, 1, 2, 4, 6, 7, \ldots \} \) corresponds to

![Young diagrams diagram](image)
Young diagrams

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\begin{array}{cccccccc}
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\text{−1} & \text{−2} & \text{−3} & \text{−4} \\
\hline
\end{array}
\]

These \( x_I \) will be the coordinates of our ambient space, partially ordered by \( I \leq J \) if \( i_k \geq j_k \) for all \( k \) (inclusion of Young diags). Unique minimum is \( I = \{1, 2, \ldots \} \).
The limit of a Plücker variety

Dual diagram

\[ \bigwedge^0 V^*_0 \leftrightarrow \bigwedge^1 V^*_0 \]
\[ \bigwedge^0 V^*_1 \leftrightarrow \bigwedge^1 V^*_1 \]

\[ \bigwedge^p V^*_{np} \leftrightarrow \bigwedge^{p+1} V^*_{n,p+1} \]
\[ \bigwedge^p V^*_{n+1,p} \]
The limit of a Plücker variety

Dual diagram

\[ \wedge^0 V^*_0 \quad \wedge^1 V^*_1 \]

\[ \wedge^0 V^*_1 \quad \wedge^1 V^*_2 \]

\( \{X_p\}_{p \geq 0} \) a Plücker variety \( \rightsquigarrow \) varieties \( X_{n,p} := X_p(V^*_n) \)
The limit of a Plücker variety

{X_p}_{p \geq 0} a Plücker variety \leadsto varieties X_{n,p} := X_p(V_{n,p}^*)
The limit of a Plücker variety

\{X_p\}_{p \geq 0} a Plücker variety $\rightsquigarrow$ varieties $X_{n,p} := X_p(V_{n,p}^*)$

$\rightsquigarrow X_\infty := \lim_{\leftarrow} X_{n,p}$ is $\text{GL}_\infty$-stable subvariety of $(\bigwedge^\infty V_\infty)^*$
The limit of a Plücker variety

Dual diagram

\[ \bigwedge^0 V^*_0 \leftarrow \bigwedge^1 V^*_0 \leftarrow \]
\[ \bigwedge^0 V^*_1 \leftarrow \bigwedge^1 V^*_1 \leftarrow \]

\{X_p\}_{p \geq 0} \text{ a Plücker variety } \leadsto \text{ varieties } X_{n,p} := X_p(V_{n,p}^*)

\leadsto X_\infty := \lim X_{n,p} \text{ is } GL_\infty\text{-stable subvariety of } (\bigwedge^{\infty/2} V_\infty)^*

Theorem (implies Main Theorem)
For bounded X, the limit X_\infty is cut out by finitely many GL_\infty-orbits of equations.
Sato’s Grassmannian

Example
The limit $\text{Gr}_\infty \subseteq (\bigwedge^\infty/2 V_\infty)^*$ of $(\text{Gr}_p)_p$ is Sato’s Grassmannian defined by polynomials $\sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0$
where $i_k = k - 1$ for $k \gg 0$ and $j_k = k + 1$ for $k \gg 0$. 
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The limit $\Gr_\infty \subseteq (\wedge^{\infty/2} V_\infty)^*$ of $(\Gr_p)_p$ is Sato’s Grassmannian defined by polynomials $\sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0$
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$\leadsto$ not finitely many $\GL_\infty$-orbits
Sato’s Grassmannian

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$\leadsto$ not finitely many $\text{GL}_\infty$-orbits

But in fact the $\text{GL}_\infty$-orbit of

$$(x_{-2,-1,3,...} \cdot x_{1,2,3,...}) - (x_{-2,1,3,...} \cdot x_{-1,2,3,...}) + (x_{-2,2,3,...} \cdot x_{-1,1,3,...})$$

defines $\text{Gr}_\infty$ set-theoretically.
Sato’s Grassmannian

Example
The limit \( \text{Gr}_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^* \) of \((\text{Gr}_p)_p\) is Sato’s Grassmannian defined by polynomials \( \sum_{i \in I} \pm x_{I-i} \cdot x_{J+i} = 0 \) where \( i_k = k - 1 \) for \( k \gg 0 \) and \( j_k = k + 1 \) for \( k \gg 0 \).

\( \leadsto \) not finitely many \( \text{GL}_\infty \)-orbits

But in fact the \( \text{GL}_\infty \)-orbit of
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\]
defines \( \text{Gr}_\infty \) set-theoretically.

Our theorems imply that also higher secant varieties of Sato’s Grassmannian are defined by finitely many \( \text{GL}_\infty \)-orbits of equations...we just don’t know which!
Polynomial time

Setting

$X$ bounded Plücker variety $\rightsquigarrow \exists n_0, p_0$ such that $\text{GL}_\infty$-orbits of equations of $X_{n_0,p_0} \subseteq \bigwedge^{p_0} V_{n_0,p_0}^*$ define $X_\infty \subseteq (\bigwedge^{\infty/2} V_\infty)^*$. 
Setting
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Shape of randomised algorithm
Input: $p, V, T \in \bigwedge^p V$
Output: $T \in X_p(V)$?
Polynomial time

Setting
X bounded Plücker variety $\leadsto \exists n_0, p_0$ such that GL$_\infty$-orbits of equations of $X_{n_0,p_0} \subseteq \wedge^{p_0} V_{n_0,p_0}^*$ define $X_\infty \subseteq (\wedge^\infty/2 V_\infty)^*$.

Shape of randomised algorithm
Input: $p, V, T \in \wedge^p V$
Output: $T \in X_p(V)$?

1. $n := \dim V - p$
2. pick random linear iso $\varphi : V \to V_{n,p}^*$
3. set $T' := (\wedge^p \varphi)T$
Polynomial time

Setting
\(X\) bounded Plücker variety \(\leadsto \exists n_0, p_0\) such that \(\text{GL}_\infty\)-orbits of equations of \(X_{n_0,p_0} \subseteq \bigwedge^{p_0} V_{n_0,p_0}^*\) define \(X_\infty \subseteq (\bigwedge^\infty/2 V_\infty)^*\).

Shape of randomised algorithm
Input: \(p, V, T \in \bigwedge^p V\)
Output: \(T \in X_p(V)\)?

1. \(n := \dim V - p\)
2. pick random linear iso \(\varphi : V \to V^*_{n,p}\)
3. set \(T' := (\bigwedge^p \varphi)T\)
4. set \(T'' := \text{image of } T' \text{ in } V^*_{n_0,p_0}\)
5. return \(T'' \in X_{n_0,p_0}\)?
A determinantal variety
\[ Y^{k,l} := \{ T \in (\bigwedge_{\infty}^2 V_{\infty})^* \mid \forall g \in \text{GL}_\infty : \]
\[ \text{image of } gT \text{ in } \bigwedge^2 V^*_{2k,2} \text{ has rank } \leq 2k \text{ and} \]
\[ \text{image of } gT \text{ in } \bigwedge^{2l} V^*_{2,2l} \text{ has rank } \leq 2l \}. \]
\[ \sim \text{ defined by orbits of two Pfaffians } \text{Pfaff}_{2k,2}, \text{Pfaff}_{2,2l} \]
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\( \rightsquigarrow \) defined by orbits of two Pfaffians \( \text{Pfaff}_{2k,2}, \text{Pfaff}_{2,2l} \)

Example with \( k = 2 \): \( \bigwedge^2 V_{4,2}^* \) has coordinates \( x_{ij} = x_i \wedge x_j \), \( i, j \in \{-4, -3, -2, -1, 1, 2\} \)

\[
\text{Pfaff}_{2k,2} = \text{Pfaff} = \begin{bmatrix}
0 & x_{-4,-3} & x_{-4,-2} & x_{-4,-1} & x_{-4,+1} & x_{-4,+2} \\
-x_{-4,-3} & 0 & x_{-3,-2} & x_{-3,-1} & x_{-3,+1} & x_{-3,+2} \\
\cdot & \cdot & 0 & x_{-2,-1} & x_{-2,+1} & x_{-2,+2} \\
\cdot & \cdot & \cdot & 0 & x_{-1,+1} & x_{-1,+2} \\
\cdot & \cdot & \cdot & \cdot & 0 & x_{+1,+2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0
\end{bmatrix}
\]
Pfaffians, continued

Example with $k = 2$:

$\text{Pfaff}_{2k,2} = \text{Pfaff}$

$$
\begin{bmatrix}
0 & x_{-4,-3} & x_{-4,-2} & x_{-4,-1} & x_{-4,+1} & x_{-4,+2} \\
-x_{-4,-3} & 0 & x_{-3,-2} & x_{-3,-1} & x_{-3,+1} & x_{-3,+2} \\
. & . & 0 & x_{-2,-1} & x_{-2,+1} & x_{-2,+2} \\
. & . & . & 0 & x_{-1,+1} & x_{-1,+2} \\
. & . & . & . & 0 & x_{+1,+2} \\
. & . & . & . & . & 0
\end{bmatrix}
$$
Example with $k = 2$: \[
Pfaff_{2k,2} = Pfaff_2(k) = x_{-4,-3} \cdot Pfaff_{2(k-1),2} + \text{terms with variables smaller than } x_{-4,-3}
\]
**Pfaffians, continued**

**Example**

with $k = 2$:

$$\text{Pfaff}_{2k,2} = \text{Pfaff}$$

$$= x_{-4,-3} \cdot \text{Pfaff}_{2(k-1),2} + \text{terms with variables smaller than } x_{-4,-3}$$

**Young diagram**

of $x_{-4,-3} = x_{-4,-3,3,4,...} \in \bigwedge^{\infty/2} V_\infty$:
Pfaffians, continued

Example with $k = 2$:

\[
Pfaff_{2k,2} = Pfaff
\]

\[
\begin{bmatrix}
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\[= x_{-4,-3} \cdot Pfaff_{2(k-1),2} + \text{terms with variables smaller than } x_{-4,-3}\]

Young diagram of $x_{-4,-3} = x_{-4,-3,3,4,...} \in \bigwedge^{\infty/2} V_{\infty}$:

Pfaffian on $\bigwedge^{2l} V_{2,2l}^*$ has largest variable
Proof

**Theorem**

$Y^{k,l}$ is a $\text{GL}_\infty$-Noetherian topological space.

(This implies the Main Theorem, since the limit of any bounded Plücker variety lies in some $Y^{k,l}$.)
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- Induction on $(k, l)$: $Y^{k,l} = Y^{k-1,l} \cup Y^{k,l-1} \cup \text{GL}_\infty Z$ where $Z = \{ T \in Y^{k,l} \mid \text{Pfaff}_{2(k-1),2}(T) \cdot \text{Pfaff}_{2,2(l-1)}(T) \neq 0 \}$
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• $Z$ is stable under

$\text{GL}(\langle \ldots, x_{-2k-2}, x_{-2k-1} \rangle) \times \text{GL}(\langle x_{2l+1}, x_{2l+2}, \ldots \rangle) =: \text{GL}_\mathbb{N} \times \text{GL}_\mathbb{N}$
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- It suffices to prove that $Z$ is $\text{GL}_\mathbb{N} \times \text{GL}_\mathbb{N}$-Noetherian.
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- It suffices to prove that $Z$ is $\text{GL}_{\mathbb{N}} \times \text{GL}_{\mathbb{N}}$-Noetherian.

- On $Z$, all variables $\geq$ or $\geq$ can be expressed in smaller ones.
Proof climax

• This leaves variables such as

```plaintext
i
2k
```

```plaintext
2l
j
```
• This leaves variables such as

\[ l, k, i \]

and hence a \( \text{GL}_N \times \text{GL}_N \)-embedding of \( Z \) into a space of \( N \)-tuples of \( \mathbb{N} \times \mathbb{N} \)-matrices, with \( i, j \) as indices.
• This leaves variables such as

\[ i \]
\[ 2k \]
\[ 2l \]
\[ j \]

• and hence a $GL_N \times GL_N$-embedding of $Z$ into a space of $N$-tuples of $N \times N$-matrices, with $i, j$ as indices.
• That space is $GL_N \times GL_N$-Noetherian, hence so is $Z!$
Proof climax

• This leaves variables such as

\[ i, 2k, 2l, j \]

and hence a \( \text{GL}_N \times \text{GL}_N \)-embedding of \( Z \) into a space of \( N \)-tuples of \( \mathbb{N} \times \mathbb{N} \)-matrices, with \( i, j \) as indices.

• That space is \( \text{GL}_N \times \text{GL}_N \)-Noetherian, hence so is \( Z \)!

Thank you!