Stabilisation in algebra, geometry, and combinatorics

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Central question

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**Topic 1** (Gaussian two-factor model)
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\(\Sigma \in \mathbb{R}^{n \times n}\), PD, is in \(X_n\) iff all \(6 \times 6\) principal submatrices are in \(X_6\).

\(X_n\) is given by polynomial eqs and ineqs; we will focus on the eqs.
Theorem [Hilbert, Math. Ann. 1890]

For a field $K$, any ideal in $K[x_1, \ldots, x_n]$ is finitely generated.

uses Dickson’s Lemma: $\alpha_1, \alpha_2, \ldots \in \mathbb{Z}_\geq 0^n \Rightarrow \exists i < j : \alpha_j - \alpha_i \in \mathbb{Z}_{\geq 0}^n$
(Non-)Noetherianity of rings

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Theorem [Cohen, J. Alg 1967; Aschenbrenner-Hillar, TAMS 2007]
For every finite set $S$, let $I_S$ be an ideal in $R_S := K[x_i \mid i \in S]$, such that any injection $\sigma : S \rightarrow T$ maps $I_S$ into $I_T$ via $x_i \mapsto x_{\sigma(i)}$. Then $I_\bullet$ is generated by $I_\emptyset, \ldots, I_{[n_0]}$ for some $n_0$.

Sym$(S)$ acts on $I_S$, and $S \mapsto R_S$ is an FI-algebra.

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same thm for $K[x_{ij} \mid i \in S, j \in [k]]$ but not for $K[x_{ij} \mid i, j \in S]$
Topological Noetherianity

**Topic 1, continued**  
[Drton-Sturmfels-Sullivant, *PTRF* 2007]

\[ X_n \subseteq \mathbb{R}^{n \times n} \]  
2-factor model, vanishing ideal \[ I_n \subseteq \mathbb{R}[x_{ij} \mid i, j \in [n]] \]
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off-diagonal $3 \times 3$-subdeterminants $\in I_n$ for $n \geq 6$

$\sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) \pi \cdot x_{12} x_{23} x_{34} x_{45} x_{51} \in I_n$ for $n \geq 5$
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These generate \( I_n \) for all \( n \geq 6 \) via injections \([6] \rightarrow [n]\).
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Replacing 2 by \(k\) we know only weaker stabilisation:


\(\forall k \exists n_0\) such that via injections \([n_0] \rightarrow [n]\) the ideal \(I_{n_0}\) generates \(I_n\) up to radical.
Instances of stabilisation

(using Noetherianity up to symmetry)
**Definition**

The *rank* of a tensor $T \in V_1 \otimes \cdots \otimes V_n$ is the minimal number of terms in any expression of $T$ as a sum of *product states* $v_1 \otimes \cdots \otimes v_n$. 
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For any fixed $k$ there is a $d$, independent of $n$ and the $V_i$, such that \{T of rank $\leq k$\} is defined by polynomials of degree $\leq d$. 
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  k & 0 & 1 & 2 \, \dagger & 3 \, \star & 4 \\
  \hline
  d & 1 & 2 & 3 \, \dagger & 4 \, \star & \geq 9 \star
\end{array}

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relevant maps from \( X(V_1, \ldots, V_n) = \{ \text{rank } \leq k \} \subseteq V_1 \otimes \cdots \otimes V_k \) into \( X(W_1, \ldots, W_n) \) or \( X(V_1, \ldots, V_{n-1} \otimes V_n) \) or \( X(V_{\pi(1)}, \ldots, V_{\pi(n)}) \)
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<tr>
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<th>1</th>
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</tr>
</thead>
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Snowden has a stabilisation result for higher syzygies for $k = 1$. 
**Second hypersimplex**

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**Theorem** [De Loera-Sturmfels-Thomas, *Combinatorica* 1995]

\( P_n \) has a Markov basis consisting of moves \( v_{ij} + v_{kl} \rightarrow v_{il} + v_{kj} \)
and \( v_{ij} \rightarrow v_{ji} \) for \( i, j, k, l \) distinct; i.e., if \( \sum_{ij} c_{ij}v_{ij} = \sum_{ij} d_{ij}v_{ij} \) with \( c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0} \), then the expressions are connected by such moves.
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**Theorem** [D-Eggermont-Krone-Leykin *Algebra & Number Th* 2016]

Any sequence $(P_n \subseteq \mathbb{Z}^n)_n$ of lattice point configurations such that $P_n = \text{Sym}(n)P_{n-1}$ for $n \gg 0$ admits a sequence $(M_n)_n$ of Markov bases such that $M_n = \text{Sym}(n)M_{n-1}$ for $n \gg 0$. 
Topic 3: Markov bases

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(Also true for \( P_n \subseteq \mathbb{Z}^{k \times n} \), considered a subset of \( \mathbb{Z}^{k \times (n+1)} \) by adding a zero column. We also have an algorithm for computing \( (M_n)_n \).)
$M$ a compact manifold
for a finite set $S$ define $C_S(M) := \{(p_i)_{i \in S} \mid p_i \neq p_j \text{ if } i \neq j\} \subseteq M^S$
for any injection $S \subseteq T$ have map $C_T(M) \to C_S(M)$
dually: $H^d(C_S(M), \mathbb{Q}) \to H^d(C_T(M), \mathbb{Q})$. 
Topic 4: homological stability

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*Among other things, this implies that the Sym($S$)-character of $H^d(C_S(M), \mathbb{Q})$ is constant for $|S| \gg 0$. 


Fix field $K$. For a finite set $S$ and a natural number $d$ let $X_{d,E} \subseteq \text{Gr}(d, K^E)$ be a Zariski-closed subset, such that:

1. deletion $\text{Gr}(d, K^E) \rightarrow \text{Gr}(d, K^{E-i})$ maps $X_{d,E}$ into $X_{d,E-i}$;
2. contraction $\text{Gr}(d, K^E) \rightarrow \text{Gr}(d-1, K^{E-i})$ maps $X_{d,E}$ to $X_{d-1,E-i}$;
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**Remark**
For finite \( K \) this is the matroid minor theorem (Geelen-Gerards-Whittle).
For infinite \( K \), the MMT does not hold, but the above might.
Grassmannians

Gr\textsubscript{k}(V) is a variety parameterising \( k \)-dimensional subspaces of \( V \). It is \textit{functorial} in \( V \), and the “Hodge dual” \( \wedge^k V \to \wedge^{n-k} V^* \) with \( \dim V = n \) maps \( \text{Gr}_k(V) \to \text{Gr}_{n-k}(V^*) \).
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A sequence $X_0, X_1, X_2, \ldots$ of rules $V \mapsto X_k(V) \subseteq \wedge^k(V)$ satisfying these two properties is called a Plücker variety.
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tangential variety, secant variety, etc.
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Theorem

[\text{D-Eggermont Crelle} 201?]

For bounded Plücker varieties, $(X_k(K^n))_{k,n-k}$ stabilises.

(For $X = \text{Gr}$, $X_\infty = \textit{Sato’s Grassmannian} \subseteq \textit{dual infinite wedge}.$)
Stabilisation in other areas

**Algebraic statistics**

*families of graphical models where the graph grows*

[Hillar-Sullivant, Takemura, Yoshida, D-Eggermont, … ]
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Commutative algebra and representation theory
higher syzygies, sequences of modules
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Thank you.