Tropical Brill-Noether theory

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The B(aker)-N(orin) game on graphs

**Requirements**
finite, undirected graph $\Gamma$
d $\geq 0$ chips
natural number $r$

**Rules**
B puts $d$ chips on $\Gamma$
N demands $\geq r_v \geq 0$ chips at $v$ with $\sum_v r_v = r$
B wins iff he can fire to meet N’s demand
Brill-Noether theorems for graphs

\[ g := e(\Gamma) - v(\Gamma) + 1 \text{ genus of } \Gamma \]
\[ \rho := g - (r + 1)(g - d + r) \]

**Conjecture (Matthew Baker)**
1. \( \rho \geq 0 \Rightarrow \text{B has a winning starting position.} \)
2. \( \rho < 0 \Rightarrow \text{B may not have one, depending on } \Gamma. \)
   \( (\forall g \exists \Gamma \forall d, r : \rho < 0 \Rightarrow \text{Brill loses.}) \)

**Theorem (Matthew Baker)**
1. is true if B may put chips at rational points of edges.
   \( (\text{uses sophisticated algebraic geometry}) \)

**Theorem (Cools-D-Payne-Robeva)**
2. is true.
   \( (\text{implies sophisticated algebraic geometry}) \)
Chip dragging on graphs

Simultaneously moving all chips along edges, with zero net movement around every cycle.

Lemma
1. Chip dragging is realisable by chip firing.
2. W.l.o.g. B drags instead of firing.

Example 1: Γ a tree
\[ \rho = g - (r+1)(g-d+r) = -(r+1)(-d+r) \]
B wins \( \iff \rho \geq 0 \iff d \geq r \)

Example 2: a hyperelliptic graph
\( d = 2, r = 1 \)
Who wins?
The B(rill)-N(oether) game on curves

Requirements
compact Riemann surface $X$
d chips
natural number $r$

Rules
B puts $d$ chips on $X$
N demands $\geq r_x \geq 0$ chips at $x$ with $\sum_x r_x = r$
B wins iff he can drag to meet N’s demand
Chip dragging on curves

Simultaneously moving chips $c$ along paths $\gamma_c : [0, 1] \to X$, such that $\sum_c \langle \omega |_{\gamma(t), \gamma'_c(t)} \rangle = 0$ for all holomorphic 1-forms $\omega$ on $X$.

Lemma

$D = \sum_c [\gamma_c(0)]$ initial position

$E = \sum_c [\gamma_c(1)]$ final position

$\iff E - D$ is divisor of meromorphic function on $X$

drag-equivalence = linear equivalence

Example: torus

only one holomorphic 1-form: $dz$

condition: $\sum_c \gamma'_c(t) = 0$

when does B win?
Dimension count

\[ \omega_1, \ldots, \omega_g \text{ basis of holomorphic 1-forms} \]
\[ \mathbf{x} = (x_1, \ldots, x_d) \in X \times \cdots \times X \]
\[ v_i \neq 0 \text{ tangent vector at } x_i \]
\[ \leadsto \text{ matrix } A_{\mathbf{x}} = (\langle \omega_i, v_j \rangle)_{ij} \in \mathbb{C}^{g \times d} \]
\[ (c_1 v_1, \ldots, c_d v_d) \text{ infinitesimal dragging direction } \Rightarrow A(c_1, \ldots, c_d)^T = 0 \]

\[ \mathbf{x} \text{ winning for } B \Rightarrow \]
\[ \text{dragging } \mathbf{x} \text{ fills } \geq r\text{-dimensional variety} \]
\[ \text{where } \ker A \text{ is } \geq r\text{-dimensional} \]

\# conditions on \( g \times d \)-matrix to have \( \geq r\)-dimensional kernel: \( r(g - d + r) \)

for B to have a winning position, “need” \( d - r(g - d + r) \geq r \)
\[ \iff \rho = g - (r + 1)(g - d + r) \geq 0 \]
Brill-Noether theorems for curves

Theorem (Meis 1960, Kempf 1971, Kleiman-Laksov 1972)
\( \rho \geq 0 \Rightarrow B \) has a winning position.

Theorem (Griffiths-Harris 1980)
1. \( \rho < 0 \Rightarrow B \) may lose, depending on \( X \).
(\( \forall g \ \exists X \ \forall d, r : \rho < 0 \Rightarrow B \) loses.)

2. \( \rho \geq 0 \) and \( X \) general
\( \Rightarrow \rho = \dim \{ \text{winning positions modulo dragging} \} \)

3. \( \rho = 0 \) and \( X \) general
\( \Rightarrow \# = \# \) standard tableaux of shape
\((r + 1) \times (g - d + r)\) with entries 1, 2, \ldots, g
Baker’s Specialisation Lemma

\( \mathcal{X} \) curve family over \( \mathbb{C}[[t]] \)
(proper, flat, regular scheme)
generic fibre \( \mathcal{X}_{\mathbb{C}((t))} \) smooth curve \( X \)
special fibre \( \mathcal{X}_{\mathbb{C}} = X_1 \cup \ldots \cup X_s \)
\( X_i \) smooth, intersections simple nodes
\( \leadsto \) dual graph \( \Gamma \) on \( \{ u_1, \ldots, u_s \} \)
(metric with edge lengths \( 1 \))
\( \leadsto \) map \( X(\mathbb{C}((t))) \rightarrow \{ u_1, \ldots, u_s \} \)

well-behaved with respect to finite extensions \( \mathbb{C}((t^{1/n}))/\mathbb{C}((t)) \)
\( \leadsto \) specialisation map \( \tau : X(\mathbb{C}\{\{t\}\}) \rightarrow \Gamma_\mathbb{Q} \)

**Theorem**
Brill wins with starting positing \( D \) on \( X(\mathbb{C}\{\{t\}\}) \)
\( \Rightarrow \) Baker wins with starting position \( \tau(D) \) on \( \Gamma_\mathbb{Q} \)
Consequences of the Specialisation Lemma

**Theorem (Conrad)**
Any graph $\Gamma$ is the dual graph of some strongly semistable model $\mathcal{X}$ whose generic fibre $X$ has genus equal to that of $\Gamma$.

**Kleiman-Laksov** ($\rho \geq 0$ implies B wins, say over $\mathbb{C}\{\{t\}\}$)
$\Rightarrow$ same statement for metric $\Gamma$.

*No combinatorial proof is known!*

**Cools-D-Payne-Robeva** ($\rho < 0 \Rightarrow$ B loses for suitable $\Gamma$)
$\Rightarrow$ Griffiths-Harris 1 (and 2, and probably 3).
Example

g = 4, d = 3, r = 1
\( g - d + r = 2 \)
r + 1 = 2, \( \rho = 0 \)

\[
\begin{array}{cc}
1 & 3 \\
2 & 4 \\
\end{array}
\] \( \rightsquigarrow \) 1, 2, 3, 2, 1

\[
\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}
\] \( \rightsquigarrow \) 1, 2, 1, 2, 1
A larger example

\[ g = 7, d = 7, r = 2 \]
\[ \Rightarrow g - d + r = 2, r + 1 = 3, \rho = 1 \]
\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 6 & 7
\end{array}
\Rightarrow (21, 31, 32, 42, 31, 31, 32, 21) \text{ lingering lattice path}

Proposition
B’s starting position \( \Rightarrow \) lingering lattice path in \( \mathbb{Z}^r \);
B wins iff path stays in chamber \( \{(x_1, \ldots, x_r) \mid x_1 > x_2 > \ldots > x_r > 0\} \).
Chips at vertices?

Theorem (van der Pol)
\( \rho \geq 0 \) and \( \Gamma \) a cactus graph
\( \Rightarrow \) B has winning positions with all chips at vertices.

Future goals:
1. Understand Kleiman-Laksov for (metric) graphs.
2. Castryck-Cools’ gonality conjecture.
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