1. Symmetric ideals according to Aschenbrenner and Hillar

We will prove the following theorem.

**Theorem 1.1.** Let \( G = \text{Sym}(\mathbb{N}) \) act on the algebra \( R = \mathbb{C}[x_0, x_1, \ldots] \) by permutations. Then any \( G \)-stable ideal \( I \) of \( R \) is finitely generated as \( G \)-stable ideal, that is, there exist finitely many \( f_1, \ldots, f_k \in I \) such that \( I \) is the smallest \( G \)-stable ideal containing \( f_1, \ldots, f_k \).

Background: Hilbert’s basis theorem says that any ideal in \( \mathbb{C}[x_0, \ldots, x_n] \) is finitely generated. But ideals in \( \mathbb{C}[x_0, x_1, \ldots] \) need not be. The above theorem says that symmetric ideals are in a sense finitely generated. We say that \( \mathbb{C}[x_0, x_1, \ldots] \) is \( G \)-Noetherian.

The proof is due to Matthias Aschenbrenner and Christopher J. Hillar. They prove something more general, but the main arguments become clear from the proof below.

**Definition 1.2.** For any map \( \pi : \mathbb{N} \to \mathbb{N} \) and \( r \in R \) we write \( \pi r \) for the image of \( r \) under the homomorphism \( R \to R \) sending \( x_i \) to \( x_{\pi i} \).

**Definition 1.3.** We define an order \( \preceq \) on monomials in \( x_0, x_1, \ldots \) as follows: it is the smallest relation on monomials satisfying \( 1 \preceq 1 \) and \( u \preceq v \Rightarrow u \preceq x_0^a \sigma v \) and \( x_0^a \sigma(u) \preceq x_0^a \sigma(v) \) for all \( u, v \) and \( 0 \leq a \leq b \). Here, as in the rest of this talk, \( \sigma : \mathbb{N} \to \mathbb{N}, i \mapsto i + 1 \).

**Definition 1.4.** For \( u \) a monomials we write \( |u| \) for the largest \( i \) such that \( x_i \) appears in \( u \). For \( u = 1 \) we write \( |u| = -\infty \).

**Lemma 1.5.** \( u \preceq v \) if and only if there is an increasing map \( \pi : \{0, \ldots, |u|\} \to \mathbb{N} \) such that \( \pi u \) divides \( v \).

**Proof.** The implication \( \Rightarrow \) follows by induction: if \( \pi \) does the trick for \( u \preceq v \), then \( \sigma \pi \), defined on \( \{0, \ldots, |u|\} \), does the trick for \( u \preceq \sigma v \), and the map defined by

\[
    i \mapsto \begin{cases} 
        \pi(i - 1) + 1 & \text{if } i > 0, \text{ and} \\
        0 & \text{if } i = 0 
    \end{cases}
\]

does the trick for \( x_0^a u \preceq x_0^a v \).

For the implication \( \Leftarrow \), from \( \pi \) one easily reconstructs a sequence of relations that deduce \( u \preceq v \) from \( 1 \preceq 1 \). \( \square \)

**Remark 1.6.** This lemma implies that \( \preceq \) is a partial order.

**Proposition 1.7.** The partial order \( \preceq \) does not have infinite antichains.

**Proof.** Suppose that there do exist infinite antichains. Then there exists an infinite never-increasing sequence

\[
    u_1, u_2, \ldots, u_n, \ldots,
\]

that is, a sequence such that \( u_i \npreceq u_j \) for all \( i < j \). Moreover, we may take such a sequence with the additional property that \( |u_n| \) is minimal among all \( u_n \) such that \( u_1, \ldots, u_n \) can be extended to an infinite never-increasing sequence.

For all \( i \) let \( a_i \) be the exponent of \( x_0 \) in \( u_i \). Now there exists an infinite sequence \( 1 \leq i_1 < i_2 < \ldots \) such that

\[
    a_{i_1} \leq a_{i_2} \leq \ldots
\]

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(take $i_1$ such that $a_{i_1}$ is minimal, then take $i_2 > i_1$ such that $a_{i_2}$ is minimal, etc.). But then consider the antichain

$$u_1, \ldots, u_{i_1-1}, u_{i_1}, u_{i_2}, \ldots.$$ 

Let $\alpha$ be the homomorphism that sends $x_{i+1}$ to $x_i$ for $i \geq 0$ and $x_0$ to 1. Consider the sequence

$$u_1, \ldots, u_{i_1-1}, \alpha(u_{i_1}), \alpha(u_{i_2}), \ldots.$$ 

By minimality of $|u_{i_1}|$, this sequence is not never-increasing. Hence either there exist $i < i_1$ and $j \geq 1$ such that

$$u_i \leq \alpha(u_{i_1}),$$

or there exist $1 \leq j \leq k$ such that

$$\alpha(u_{i_j}) \leq \alpha(u_{i_k}).$$

But in the first case we have

$$u_i \leq u_{i_1}$$

by the first inductive property of $\leq$, and in the second case we have

$$u_{i_j} \leq u_{i_k}$$

by the second inductive property and the fact that $a_{i_j} \leq a_{i_k}$. We thus arrive at a contradiction, hence the proposition is proved. \hfill \Box

Now we can prove the theorem.

**Proof of Theorem 1.1.** Let $I$ be a $G$-stable ideal. To any $f \in R$ we associate its leading monomial $\text{lm}(f)$ in the lexicographic order, where $x_1 < x_2 < \ldots$. So for instance $x_2^3 < x_1^2 x_2 < x_3$, and $x_3$ is the leading monomial in $x_2^3 + x_1 x_2 + x_3$. Now consider the set $M$ of all $\leq$-minimal elements of the set $\{\text{lm}(f) \mid f \in I\}$. This is an antichain by definition, hence finite by the proposition. Hence there exist (monic) $f_1, \ldots, f_k \in I$ such that $M = \{\text{lm}(f_1), \ldots, \text{lm}(f_k)\}$. We claim that $I$ equals the smallest $G$-stable ideal $J$ containing $f_1, \ldots, f_k$.

Indeed, suppose that $I$ contains a (monic) counterexample $f \not\in J$. We may assume that $\text{lm}(f)$ lexicographically minimal among counterexamples (since the lexicographic order is a well-order). By construction, there exists an $i$ such that $\text{lm}(f_i) \leq \text{lm}(f)$. Set $n := |\text{lm}(f_i)|$ and let $\pi : \{1, \ldots, n\} \to \mathbb{N}$ be increasing such that $\pi(\text{lm}(f_i))|\text{lm}(f)$; say $\text{lm}(f) = u \pi(\text{lm}(f_i))$. Then $\pi(f_i) \in J$ by $G$-stability, and

$$f' := f - u \pi(f_i) \not\in J.$$

We claim that the $\text{lm}(f')$ is lexicographically smaller than $\text{lm}(f)$, contradicting the minimality of the latter. But this is clear from $\text{lm}(\pi(f_i)) = \pi(\text{lm}(f_i))$, so that

$$\text{lm}(u \pi(f_i)) = u \pi(\text{lm}(f_i)) = \text{lm}(f).$$

\hfill \Box

### 2. $G$-Noetherianity of Some Modules

Let the group $G = \text{Sym}(\mathbb{N})$ act on the ring $R = K[y_{ij} \mid i \neq j]$ by permuting the indices simultaneously. It is easy to see that this ring is not $G$-Noetherian. However, let $R_{\leq d}$ denote the $G$-module of polynomials of degree at most $d$.

**Proposition 2.1.** The $G$-module $R_{\leq d}$ is Noetherian, i.e., every $G$-submodule of it is finitely generated.
Proof. We proceed as above: we define two partial orders on monomials in $R$. The first one has $u \preceq v$ if and only if there is a strictly increasing map $\pi : \{1, \ldots, |u|\} \to \mathbb{N}$ such that $\pi u = v$. Here $|u|$ denotes the maximum among all indices appearing in variables in $u$. The second order is lexicographic, where the largest index of a variable is most significant, and for definiteness $y_{ij} < y_{ji}$ if $i < j$. So for instance $y_{31} > y_{13}$ and $y_{21} = y_{21}y_{12} > y_{12}^3$.

We claim that the monomials in $R_{\leq d}$ do not contain an infinite antichain with respect to $\preceq$. Indeed, if such an antichain exists, then since there are only finitely many $G$-orbits of monomials in $R_{\leq d}$, there exists an antichain $C$ contained in some $G$-orbit. Fix $u$ in this $G$-orbit for which the indices appearing in its variables are precisely the numbers $1, \ldots, n$. For any element $v$ of $Gu$ construct a monomial $m(v)$ in the variables $x_1, x_2, \ldots$ as follows: let $\pi_v$ be a bijection from $\{1, \ldots, n\}$ to the set of indices appearing in $v$, such that $\pi_v u = v$. Then set $m(v) := \pi_v(x_1^1 x_2^2 \cdots x_n^n)$. In particular, if we choose $\pi_u = \text{id}$, then $m(u) = x_1 x_2^2 \cdots x_n^n$. Now $m$ is an injection from $Gu$ to monomials in $x_1, x_2, \ldots$, hence it maps $C$ to an infinite set. This cannot be an antichain in the order $\preceq$ on monomials in the $x_i$ introduced earlier, hence $m(v) \preceq m(w)$ for some $v, w \in C$. Hence there exists an increasing map $\tau : \{1, \ldots, |v|\} \to \mathbb{N}$ such that $\tau m(v) = m(w)$. But then also $\tau v = w$, hence $v \preceq w$.

Now let $P$ be a $G$-submodule of $R_{\leq d}$. Denote by $M$ the set of all $\preceq$-minimal elements of $\{\text{lm}(f) \mid f \in P\}$. Then $M$ is an antichain, and finite by the above. Hence there exist (monic) $f_1, \ldots, f_k \in M$ such that $M = \{\text{lm}(f_1), \ldots, \text{lm}(f_k)\}$. We claim that $P$ equals the $G$-module $Q$ generated by the $f_i$.

Indeed, suppose that $P$ contains a (monic) counterexample $f \notin Q$. We may assume that $\text{lm}(f)$ lexicographically minimal among counterexamples (since the lexicographic order is a well-order). By construction, there exists an $i$ such that $\text{lm}(f_i) \leq \text{lm}(f)$. Set $n := |\text{lm}(f_i)|$ and let $\tau : \{1, \ldots, n\} \to \mathbb{N}$ be increasing such that $\tau(\text{lm}(f_i)) = \text{lm}(f)$. Then $\tau(f_i) \in Q$ by $G$-stability, and $f' := f - \tau(f_i) \notin Q$.

We claim that the $\text{lm}(f')$ is lexicographically smaller than $\text{lm}(f)$, contradicting the minimality of the latter. But this is clear from $\text{lm}(\tau(f_i)) = \tau(\text{lm}(f_i)) = \text{lm}(f)$. □