Tropical Algebraic Groups ?!

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or Google “Tropical Algebraic Groups” and find me.
Algebraic Groups

Definition. *algebraic group* $G$: variety + compatible group structure.

Lemma. $G$ affine $\Rightarrow$ closed subgroup of some $GL_n$.

This project: only affine groups
(as opposed to Abelian varieties).

Example.
$SL_n = \{ g \in M_n \mid \det(g) = 1 \}$
$O_n = \{ g \in M_n \mid g^T g = I \}$
$U_n = \{ g \in M_n \mid g_{ii} = 1, g_{ij} = 0 \text{ for all } i > j \}$

Algebraic groups describe symmetries of beautiful varieties such as Segre-Veronese embeddings of projective spaces and Grassmannians.
Goal

Goal of the project: tropicalise affine algebraic groups and their actions on varieties—but how? Tropicalisation depends on coordinates!

Remark.
Berkovich space $\mathcal{B}G$ of $G = “all possible tropicalisations of G”$ doesn’t. But $\mathcal{B}(G \times G) \neq \mathcal{B}(G) \times \mathcal{B}(G)$, so Berkovichisation of multiplication is not a map $\mathcal{B}(G) \times \mathcal{B}(G) \rightarrow \mathcal{B}(G)$.

Provisional remedy: work in nice coordinates.
A nice example

$$\mathcal{T}_{\text{SL}}_n = \{ w \in M_n(\mathbb{R}) \mid \text{minimal weight of a perfect matching is } \leq 0 \text{ and attained twice if } < 0 \}$$

Example.

```
\begin{array}{ccc}
0 & 1 & 5 \\
1 & 7 & -6 \\
5 & -2 & 0 \\
\end{array}
```

tdet $= 1 + 1 - 2 = 0$

Theorem (Egerváry, 1931).

$$\exists a_1, \ldots, a_n, b_1, \ldots, b_n : w_{ij} \geq a_i + b_j$$
with $=$ on the edges of some $w$- minimal perfect matching.

$\Rightarrow$ description of full- dimensional cones of $\mathcal{T}_{\text{SL}}_n$

Proposition. $\mathcal{T}_{\text{SL}}_n$ is a monoid under tropical matrix multiplication.

Proof.
An example that ought to be nice

\[ O_n = \{ g \in M_n \mid g^T g = I \}; \text{ what is } \mathcal{T}O_n?? \]

Challenges:

- a tropical basis?
  (equations above, \( \det(g) = \pm 1, g_{ij} = \det(g_{[n]-i,[n]-j}) \))
- is it a monoid?

Definition. \( D = (d_{ij}) \) is metric if \( d_{ii} = 0, d_{ij} = d_{ji}, d_{ij} + d_{jk} \geq d_{ik} \).

Proposition. \( \mathcal{T}O_n \) contains the cone of all metric matrices.

Proof. Use the Lie algebra of \( O_n \). Consider

\[
\exp\begin{bmatrix}
0 & x_{12} & \cdots & x_{1n} \\
-x_{12} & 0 & \ddots & \\
\vdots & \ddots & \ddots & x_{n-1,n} \\
-x_{1n} & \ldots & -x_{n-1,n} & 0
\end{bmatrix}
\]

where \( x_{ij} \) has valuation \( d_{ij} \)
Total positivity

Example. \( \text{GL}_n^+(\mathbb{R}_{\geq 0}) \) := monoid in \( \text{GL}_n(\mathbb{R}) \) generated by

- diagonal matrices with positive diagonal entries, and
- the 1- parameter groups \( x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} \):

\[
x_1(t) = \begin{bmatrix} 1 & t \\ 1 & \ddots \\ & & 1 \end{bmatrix} \text{ with } t > 0, y_1 = x_1^T, \text{ etc.}
\]

Theorem (classical).
\( \text{GL}_n^+(\mathbb{R}_{\geq 0}) = \{ g \in \text{GL}_n(\mathbb{R}) \mid \text{all determinants of square submatrices of } g \text{ are } \geq 0 \}. \)
Example (continued). $\mathcal{T}^+\text{GL}_n := \text{submonoid of } M_n(\overline{\mathbb{R}})$ generated by

- diagonal matrices without $\infty$ on the diagonal
- the $x_i(a), y_i(a)$ with $a \in \overline{\mathbb{R}}$.

Proposition.

$\mathcal{T}^+\text{GL}_n = \{ w \in M_n(\overline{\mathbb{R}}) \mid w_{ij} + w_{kl} \leq w_{il} + w_{kj} \text{ if } i < k, j < l \}$,

Proof of $\subseteq$.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\hline
& & & \\
& & & \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
d1 & d2 & d3 & d4 \\
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array}
\]

1342 + 1233 < 1342 + 1234

\[
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
\end{array}
\]

d1 d2 d3 d4 diagonal

$w_{12} + w_{33} < w_{13} + w_{32}$

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\hline
\end{array}
\]

Proof of $\subseteq$. \qed
Positive monoid

$F$: a zero-sum-free semifield
$\Gamma$: a Dynkin diagram

**Definition.** $G^+(F) :=$ monoid presented by

- generators: $x_i(a), t_i(b), y_i(c)$ for vertices $i$ of $\Gamma$, $a, c \in F$, $b \in F^*$.
- relations mimicking those in $\text{GL}_n^+$, e.g.

\[
\begin{align*}
    x_i(0) & = 1 & x_i(a)x_i(b) & = x_i(a + b) \\
    t_i(ab) & = t_i(a)t_i(b) & t_i(1) & = 1 \\
    x_i(a)x_j(b)x_i(c) & = x_j(bc/(a + c))x_i(a + c)x_j(ab/(a + c)) \text{ for } i \not\sim j \\
    y_i(b)x_i(a) & = x_i(a/(ab + 1))t_i(1/(ab + 1))y_i(b/(ab + 1)) \\
\end{align*}
\]

*Lusztig variety* (Berenstein- Fomin- Zelevinsky).
Positive Monoid

Remark. $G^+$ is a functor from zero-sum-free semifields to monoids.

Example.

$y_i(b)x_i(a) = x_i(a/(ab + 1))t_i(1/(ab + 1))y_i(b/(ab + 1))$ comes from

$$
\begin{bmatrix}
1 & 0 \\
b & 1
\end{bmatrix}
\begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{a}{ab+1} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{ab+1} & 0 \\
0 & ab + 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\frac{b}{ab+1} & 1
\end{bmatrix}
$$

Example.

$\Gamma =$ path of length $n - 1$ and $F = \mathbb{R}_{\geq 0}$ gives $GL^+_n$,

but for $F = \overline{\mathbb{R}}$ the map $G(\mathbb{R})^+ \to M_n(\mathbb{R})$ is not injective.

Proposition (Lusztig).

$\exists$ finite word $x_{i_1} \cdots x_{i_l}t_1 \cdots t_ny_{i_1} \cdots y_{i_l}$ such that $F^l \times (F^*)^n \times F^l \to G^+(F)$ surjective.
Representations of the positive monoid

$G$: simply connected group over $\mathbb{C}$ with diagram $\Gamma$

Every $G$-representation $V$ of $G$ has a canonical basis.

In these coordinates the generators $x_i, t_i, y_i$ make sense over any zero-sum-free semifield $F$.

**Observation:** image of $G^+(\overline{\mathbb{R}}) \rightarrow M_n(\overline{\mathbb{R}})$ is contained in $\mathcal{T}G$ (take $F = \text{Puiseux series with positive leading coefficient and use functoriality}$).

**Sample Questions:**

- Is Speyer-Williams’ totally positive Grassmannian one orbit under $GL_n^+(\overline{\mathbb{R}})$?

- $\exists$ representation $V$ faithful for all $F$? (I think yes; product of fundamental representations?)

- Describe the polyhedral complexes that are the images of $G^+(\overline{\mathbb{R}})$ in representations.

$\Rightarrow$ these should be our “nice coordinates”
A “naive” approach

Definition. An additive tropical 1-PSM is a tropical map $x : \mathbb{R} \rightarrow M_n(\mathbb{R})$ such that $x(\infty) = I$ and $x(a + b) = x(a)x(b)$.

Lemma. $x$ is upper triangular up to a permutation.

Questions:
- Given finitely many 1-PSM’s $x_i$, what monoid do they generate?
- $\exists$ finite word in the $x_i$ surjective on the monoid?
- Is the monoid pure?

Proposition. All $x_i$ simultaneously upper-triangular $\Rightarrow \exists$ a finite word surjective on the monoid.