A tropical approach to secant varieties

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A vignette

\[ C = \text{Cone over Segre}(\text{Veronese}_2(\mathbb{P}^1) \times \text{Veronese}_2(\mathbb{P}^1) \times \text{Veronese}_2(\mathbb{P}^1)) \]

\[ kC = \{ p_1 + \ldots + p_k \mid p_i \in C \} \]

\[ \dim kC = (\geq)4, 8, 12, 16, 20, 24, 26, 27 \]
A jigsaw puzzle

d = 3
Solutions for $d = 2, \ldots, 6$
A not so puzzling result

**Theorem 1.** $d$ is non-defective unless

- $d = 2$ (only 1 triangle fits), or
- $d = 4$ (only 4 triangles fit).

Proof by induction.
Polynomial interpolation

Given: $k$ generic points $p_1, \ldots, p_k \in \mathbb{P}^2$ and $d \in \mathbb{N}$.

**Theorem 2** (Hirschowitz, 1985).

$$\dim \{ \text{homogeneous polynomials of degree } d \text{ having singularities at all } p_i \} = \max\{0, \binom{d+2}{2} - 3k\}$$ unless

- $(d, k) = (2, 2)$: dimension $= 1$, or
- $(d, k) = (4, 5)$: dimension $= 1$.

Proof by induction with the *Horace method*. 
A tautology?

Theorem 3. \( \text{Theorem 1} \Rightarrow \text{Theorem 2} \)

Proof: *tropical geometry*!
Secant varieties

$C$ a closed cone in a $K$-space $V$, $k \in \mathbb{N}$

$kC := \{v_1 + \ldots + v_k \mid v_i \in C\}$,

the $k$-th secant variety of $C$


**Example 4.**

- $C_1 = \text{rank} \leq 1$ matrices in $V_1 = \mathbb{M}_m$
  $kC_1 = \text{rank} \leq k$ matrices

- $C_2 = \{z_1 \wedge z_2\} \subseteq V_2 := \bigwedge^2 \mathbb{K}^m$
  cone over Grassmannian of 2-spaces in $\mathbb{K}^m$
  $kC_2 = \text{skew-symmetric matrices of rank} \leq 2k$

- $C_3 = \text{cone over Grassmannian of isotropic 2-spaces in } \mathbb{K}^m$
  $2C_3$ and $3C_3$ are complicated
  $kC_3 = kC_2$ for $k \geq 4$ (Baur and Draisma, 2004)
Non-defectiveness

Note: $\dim kC \leq \min\{k \dim C, \dim V\}$, the expected dimension.

**Definition 5.**
$kC$ is non-defective if $\dim kC$ is as expected.
$C$ is non-defective if all $kC$ are.

Many $C$'s are non-defective, but hard to prove so!
Relation to polynomial interpolation

\[ V = K^m \]
\[ C := \{ v^d \} \subseteq S^d V \]
\[ \dim kC =? \]

**Lemma 6** (Terracini, 1911). *For \( v_1, \ldots, v_k \in V \) generic*
\[ \dim kC = \dim T_{v_1}C + \ldots + \dim T_{v_k}C. \]

Lasker, 1904:
\[ T_{v_i}C = \{ f \in S^d(V^*) \mid f \text{ is singular in } [v_i] \in \mathbb{P}V \}^0 \]
so
\[ \dim kC = \operatorname{codim} \{ \text{hom. pols. of degree } d \text{ singular in all } [v_i] \}. \]

Alexander and Hirschowitz, 1995: \( \dim kC \) for all \( k, d, m \).
Secant dimensions of other classes of cones (e.g., Segre products and Grassmannians) still unknown!
Algebraic and tropical geometry

A rough guide:

<table>
<thead>
<tr>
<th>algebraic geometry</th>
<th>tropical geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>embedded affine variety $X \subseteq K^n$</td>
<td>polyhedral complex $T(X) \subseteq \overline{\mathbb{R}}^n$</td>
</tr>
<tr>
<td>polynomial map $f$</td>
<td>piecewise linear map $T(f)$</td>
</tr>
<tr>
<td>dim $X$</td>
<td>dim $T(X)$</td>
</tr>
</tbody>
</table>

Strategy: $\dim T(kC) = k \dim C \Rightarrow kC$ is non-defective.

But $kC$ is not known! Solution:

- parametrise $h : K^m \to C \subseteq V$
- tropicalise $f : (K^m)^k \to kC$, $(z_1, \ldots, z_k) \mapsto h(z_1) + \ldots + h(z_k)$
- compute $\text{rk } dT(f)$ at a good point
A simplified theorem

\[ h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n \text{ parametrisation} \]
Assume: each \( h_b = c_b x^{\alpha_b} \neq 0 \) (1 term).

Choose a 2-norm on \( \mathbb{R}^m \).
For \( v = (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k \) set \( \text{Vor}_i(v) := \{ \alpha_b \text{ in Voronoi cell of } v_i \} \).

**Theorem 7** (Draisma, 2006). \( \sum_i (1 + \dim \text{Aff}_{\mathbb{R}} \text{Vor}_i(v)) \leq \dim kC \).

Lower bound=\(2+1+1\)
Finally, a proof!

Proof of Theorem 3.
$C =$ Veronese cone
$h : (x_1, x_2, x_3) \mapsto (x_1 e_1 + x_2 e_2 + x_3 e_3)^d \in S^d(K^3)$
Combine theorem 7 and the jigsaw puzzle.

Generalisation to higher dimensions?
Tropical geometry

\( K \) a field with non-archimedean valuation \( v : K \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \)

\( X \) affine variety over \( K \)

\( \bar{x} := (\bar{x}_1, \ldots, \bar{x}_n) \) generators of \( K[X] \)

**Definition 8** (Tropicalisation of a variety).

\[
T_{\bar{x}}(X) := \left\{ (v'(\bar{x}_1), \ldots, v'(\bar{x}_n)) \big| \right. \\
v' : K[X] \rightarrow \overline{\mathbb{R}} \text{ valuation extending } v \}
\]

Depends on generators/embedding!

**Example 9.**

\( X = \mathbb{A}^1, \ K[X] = K[x] \),

\( \bar{x}_1 = x, \ \bar{x}_2 = x + 1 \)
More concretely

$L/K$ algebraically closed, complete extension with $v(L) = \overline{\mathbb{R}}$

$I$ ideal of $X$ in $K[x] = K[x_1, \ldots, x_n]$

For $w \in \overline{\mathbb{R}}^b$, $P = \sum_{\alpha} c_{\alpha}x^\alpha \in K[x]$:

$\text{wt}_w P := \min_{\alpha} v(c_{\alpha}) + w \cdot \alpha$

$\text{in}_w P := \text{terms of minimal weight}$

**Theorem 10** (Einsiedler-Kapranov-Lind, Speyer-Sturmfels, see also D.).

The following are equal:

- $v(X(L))$
- $T_{\overline{x}}(X)$
- $\{w \in \mathbb{R}^n | \text{in}_w f \text{ is not monomial for any } f \in I\}$

**Theorem 11** (Bieri-Groves, Sturmfels). $T_{\overline{x}}(X)$ is a polyhedral complex, pure of dimension $\dim X$ if $X$ is irreducible.
Definition 12 (Tropicalisation of polynomials).

\[ T(P)(w) := wt_w P \]
\[ T(h_1, \ldots, h_b) := (T(h_1), \ldots, T(h_b)) \]

Example 13.

\[ P = cx_1^2 + x_2 \]
\[ T(P) = \min\{c + 2w_1, w_2\} \]

Lemma 14. Given \( h : K^m \to C \)
and \( f(z_1, \ldots, z_k) := h(z_1) + \ldots + h(z_k) \),
\[ T(f) \text{ maps } (\mathbb{R}^m)^k \text{ into } T(kC). \]

Find a point where

- \( T(f) \) is linear and
- \( dT(f) \) has maximal rank.

This leads to Theorem 7.
Generalisation of Theorem 7

Optimisation problem:

Given

\( k \in \mathbb{N} \)

\( A = (A_1, \ldots, A_n) \) list of finite subsets of \( \mathbb{R}^n \)

**Optimisation domain**

\( k \)-tuples \( l = (l_1, \ldots, l_k) \) of affine linear functions on \( \mathbb{R}^n \)

**Objective function**

\[
\sum_{i=1}^{k}(1 + \dim \text{Aff}_{\mathbb{R}} C_i(l))
\]

where \( C_i = \bigcup_{b=1}^{n}\{\alpha \in A_b \mid f_i(\alpha) < f_j(\beta) \text{ for all } (\beta, j) \in A_b \times \{1, \ldots, k\}\} \)

**Optimal value** \( \text{AP}^*(A, k) \)

**Theorem 15.** \( h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n \) parametrisation

\( A_b \subseteq \mathbb{N}^m \) support of \( h_b \)

then \( \text{AP}^*(A, k) \leq \dim kC \)
Some results (with Karin Baur)

Non-degenerate:

1. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except (even, 2)
2. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except (even, 1, 1)
3. Segre embedding of $(\mathbb{P}^1)^6$ (cells for $k = 9$: 8 disjoint Hamming balls of radius 1 and one cell in the middle)

Almost done:

1. $\mathbb{P}^1 \times \mathbb{P}^2$
2. \{$\text{flags point} \subseteq \text{line} \subseteq \mathbb{P}^2$\} probably all non-defective except those of highest weight $\omega_1 + \omega_2$ (adjoint representation) or $2\omega_1 + 2\omega_2$ needs Theorem 15 rather than 7

Bold conjecture: the lower bound is always correct for Segre-Veronese embeddings.
picture suggests: S-V embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2, 2, 2)$ has defective $7C$. Indeed!

Minimal orbit in representation of $SL_3$ of highest weight $(5, 1)$ is non-defective.
Conclusion

**Non-defectiveness** often provable by optimising a strange polyhedral-combinatoric objective function

**Hope** a point in $T(kC)$ with full-dimensional neighbourhood gives restrictions on the ideal of $kC$. Sufficient to settle one or two more cases of GSS?

**Segre-Veronese** is the given bound always correct?

**Other minimal orbits** Smallest flag variety doable with a trick: reduce all $A_b$ in Theorem 15 to singletons, and use Voronoi-variant. In general: which parametrisation to use? (Littelmann-Bernstein-Zelevinsky polytopes?)
Reading

- Catalisano-Geramita-Gimigliano: some secant dimensions of Grassmannians, non-defectiveness of most $k(\mathbb{P}^1)^d$, defectiveness of some unbalanced Segre products and Segre-Veronese products.

- Sturmfels-Sullivant, Miranda-Dumitrescu, 200*: degeneration approach for secant dimensions

- Landsberg-Weyman, 2006: equations for certain secant varieties of Segre products (GSS conjectures!)

- Baur-Draisma-de Graaf, 2006: GAP-program for computing secant dimensions of minimal orbits

- Draisma, 2006: A tropical approach to secant dimensions math.AG/0605345 (includes intro to tropical geometry!)

Hope this was not too non-linear