A tropical approach to secant varieties

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Typical example: polynomial interpolation in two variables

Set-up:
d \in \mathbb{N}
p_1, \ldots, p_k \text{ general points in } \mathbb{C}^2
\text{codim}\{ f \in \mathbb{C}[x, y]_{\leq d} \mid \forall i : f(p_i) = f_x(p_i) = f_y(p_i) = 0 \} = ??
expect: \min\{3k, \binom{d+2}{2} \} \text{ (upper bound)}

Hirschowitz (1985):
correct, unless (d, k) = (2, 2) or (d, k) = (4, 5) (1 instead of 0)

D (2006): new proof using tropical geometry, paper and scissors

Alexander and Hirschowitz: more variables (1995)
Also doable tropically??
Tropical Semiring

\[ \overline{\mathbb{R}} := \mathbb{R} \cup \{ \infty \} \]
\[ a \oplus b := \min \{ a, b \} \]
\[ a \odot b := a + b \]

\[ \infty \oplus b = b \]
\[ 0 \odot b = b \]
\[ a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c) \]


**Tropical polynomials**

\( \alpha, \beta, \ldots \in \mathbb{N}^n \)
\( a, b, \ldots \in \overline{\mathbb{R}} \)

the map \( \overline{\mathbb{R}}^n \to \overline{\mathbb{R}} \)
\( \xi \mapsto a \odot \bigodot_{i=1}^n \xi_i^{\odot \alpha_i} + b \odot \bigodot_{i=1}^n \xi_i^{\odot \beta_i} \)

\[ = \min\{ a + \langle \xi, \alpha \rangle, b + \langle \xi, \beta \rangle, \ldots \} \] is a **tropical polynomial**

**Example**

\( A \in \overline{\mathbb{R}}^{n \times n} \)

\( \text{tdet}(A) := \bigoplus_{\pi \in S_n} a_{\pi(1),1} \odot a_{\pi(2),2} \odot \cdots \odot a_{\pi(n),n} \) **tropical determinant**

minimal weight matching in \( K_{n,n} \) with edge weights \( a_{ij} \)

\[ \text{tdet} = 1+5+5 \]
Tropical geometry

Set-up:

$K$ field

$v : K \rightarrow \mathbb{R}$ non-Archimedean valuation, that is, $v^{-1}(\infty) = \{0\}$, $v(ab) = v(a) \odot v(b)$, and $v(a + b) \geq v(a) \oplus v(b)$

e.g. $K =$Laurent series and $v =$multiplicity of $0$ as a zero
technical conditions on $(K, v)$

$X \subseteq K^n$ given by polynomial equations

$\mapsto TX := \{v(x) = (v(x_1), \ldots , v(x_n)) \mid x \in X\}$

tropicalisation of $X$

depends on coordinates!
Codimension one varieties

$X$ zero set of one polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$

$\mathcal{T} f(\xi) := \min_{\alpha \in \mathbb{N}^n} (v(c_\alpha) + \langle \xi, \alpha \rangle)$ tropicalisation of $f$

**Theorem 1** (Einsiedler–Kapranov–Lind).

$\mathcal{T}X = \{ \xi \in \mathbb{R}^n \mid \mathcal{T} f \text{ not linear at } \xi \}$

$\rightsquigarrow$ tropical hypersurfaces are polyhedral complexes!

**Example:**

$f = x_1 + x_2 - 1 \text{ (line)}$

$\mathcal{T} f = \min \{ \xi_1, \xi_2, 0 \}$
Plane Curves: conics
Mikhalkin (re)computed the number of classical degree $d$, genus $g$ plane curves through $3d + g - 1$ general points: count tropical such curves, each with a certain multiplicity.
Arbitrary codimension

$I$ the ideal of $X \subseteq K^n$

**Theorem 2** (EKL 2004, SS 2003, see also D 2006).

\[
\mathcal{T} X = \{ w \in \mathbb{R}^n | \forall f \in I : \mathcal{T} f \text{ not linear at } w \}
\]

**Theorem 3** (Bogart–Jensen–Speyer–Sturmfels–Thomas (2005)).

\[ \exists \text{ finite subset of } I \text{ for which previous theorem is true} \]

\[ \leadsto \text{tropical basis (hard to compute!)} \]

\[ \leadsto \mathcal{T} X \text{ is a polyhedral complex} \]

**Theorem 4** (Bieri–Groves (1985), Sturmfels).

\[ X \text{ irreducible of dimension } d \Rightarrow \dim \mathcal{T} X = d \]
Example: $\mathcal{T} SL_n$ and $\mathcal{T} O_n$

Codimension one:
$\mathcal{T} SL_n = \{ A \in \mathbb{R}^{n \times n} \mid \text{tdet } A \leq 0 \text{ and if } < 0 \text{ then attained at least twice} \}$
monoid under tropical matrix multiplication

Higher codimension:
$O_n := \{ g \mid g^t g = I \}$
$\mathcal{T} O_n = ??$
closed under tropical matrix multiplication?
not sufficient to tropicalise the $n^2$ defining equations
Observation (D-McAllister 2006):
$\mathcal{T} O_n \supset \{ n\text{-point metrics} \}$ (full-dimensional cone)
tropical matrix multiplication $= \text{composition of metrics}$
Secant varieties

$C$ a closed cone in a $K$-space $V$, $k \in \mathbb{N}$

$kC := \{v_1 + \ldots + v_k \mid v_i \in C\}$,

the $k$-th secant variety of $C$


Example 5.

- $C_1 = \text{rank} \leq 1$ matrices in $V_1 = M_m$
  $kC_1 = \text{rank} \leq k$ matrices

- $C_2 = \{z_1 \wedge z_2\} \subseteq V_2 := \bigwedge^2 K^m$
  cone over Grassmannian of 2-spaces in $K^m$
  $kC_2 = \text{skew-symmetric matrices of rank} \leq 2k$

- $C_3 = \text{cone over Grassmannian of isotropic 2-spaces in } K^m$
  2$C_3$ and 3$C_3$ are complicated
  $kC_3 = kC_2$ for $k \geq 4$ (Baur and Draisma, 2004)
Non-defectiveness

Note: $\dim kC \leq \min\{k \dim C, \dim V\}$, the *expected dimension*.

**Definition 6.**

$kC$ is *non-defective* if $\dim kC$ is as expected.

$C$ is non-defective if all $kC$ are.

Many $C$'s are non-defective, but hard to prove so!
Minimal orbits: interesting cones

$V$ irrep of complex algebraic group $G$
$v \in V$ highest weight vector
$C := Gv \cup \{0\} \subseteq V$

all examples so far were of this form
dimensions of $kC$, $k = 1, 2, \ldots$ largely unknown!
(except $V = S^d(C^n)$ for $G = GL_n$—Alexander & Hirschowitz (1995))

widely open: tensor products, Grassmannians, etc.
Tropical strategy for proving $kC$ non-defective

Recall:

<table>
<thead>
<tr>
<th>algebraic geometry</th>
<th>tropical geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>embedded affine variety $X \subseteq K^n$</td>
<td>polyhedral complex $\mathcal{T}(X) \subseteq \mathbb{R}^n$</td>
</tr>
<tr>
<td>polynomial map $f$</td>
<td>piecewise linear map $\mathcal{T}(f)$</td>
</tr>
<tr>
<td>dim $X$</td>
<td>dim $\mathcal{T}(X)$</td>
</tr>
</tbody>
</table>

Strategy: prove $\dim \mathcal{T}(kC) = k \dim C$; then $kC$ is non-defective.
But $kC$ not known, let alone $\mathcal{T}(kC)$!
Proposal:

- parametrise $h : K^m \rightarrow C \subseteq V$
- tropicalise $f : (K^m)^k \rightarrow kC$, $(z_1, \ldots, z_k) \mapsto h(z_1) + \ldots + h(z_k)$
- compute $\text{rk } d\mathcal{T}(f)$ at a good point $\leadsto$ lower bound on $\dim \mathcal{T}kC$
A (simplified) theorem

\( h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n \) parametrisation
assume each \( h_b = c_b x^{\alpha_b} \neq 0 \) (1 term)

for \( l = (l_1, \ldots, l_k) \) \( k \) affine-linear functions on \( \mathbb{R}^m \) set
\( C_i(l) := \{ \alpha_b | l_i(\alpha) < l_j(\alpha) \text{ for all } j \neq i \} \)

**Theorem 7** (Draisma, 2006).

\[ \dim kC \geq \sum_i (1 + \dim \text{Aff}_\mathbb{R} C_i(l)) \]

Lower bound = 3 + 2 + 1
Funny optimisation problem

\[ A \subseteq \mathbb{R}^n \text{ finite, } k \in \mathbb{N} \]

Maximise \( \sum_i (1 + \dim \text{Aff}_\mathbb{R} C_i(l)) =: * \)

over all \( l = (l_1, \ldots, l_k) \), each \( l_i \) affine-linear

Corollary 8.

\( A = \{ \alpha_b \mid b \} \) exponents of monomials in parametrisation

draw \( A \) on \( m \)-dimensional paper

cut paper into \( k \) pieces

compute sg. like *

\( \leadsto \) lower bound on \( \dim kC \)
Example from beginning:

\[ C = \text{Veronese cone} \]

\[ h : (x_1, x_2, x_3) \mapsto (x_1 e_1 + x_2 e_2 + x_3 e_3)^d \in S^d(C^3) \]

satisfies conditions of theorem \( \rightsquigarrow \) paper-and-scissors lower bound

Generalisation to higher dimensions?
Some results (with Karin Baur)

Non-degenerate:

1. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 2)$
2. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 1, 1)$
3. Segre embedding of $(\mathbb{P}^1)^6$ (cells for $k = 9$: 8 disjoint Hamming balls of radius 1 and one cell in the middle)

Almost done:

1. $\mathbb{P}^1 \times \mathbb{P}^2$
2. $\{\text{flags point } \subseteq \text{ line } \subseteq \mathbb{P}^2\}$ probably all non-defective except those of highest weight $\omega_1 + \omega_2$ (adjoint representation) or $2\omega_1 + 2\omega_2$ needs generalisation of Theorem 7

Conjecture: the lower bound always gives correct dimension for Segre-Veronese embeddings.
picture suggests: S-V embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2, 2, 2)$ has defective $7C$. Indeed!

Minimal orbit in representation of $\text{SL}_3$ of highest weight $(5, 1)$ is non-defective.
Conclusions

Non-defectiveness often provable by optimising a strange polyhedral-combinatoric objective function

Hope a point in $T(kC)$ with full-dimensional neighbourhood gives restrictions on the ideal of $kC$. Sufficient to settle one or two more cases of GSS?

Segre-Veronese is the given bound always correct?

Other minimal orbits Smallest flag variety has non-monomial parametrisation, but doable with a trick. In general: which parametrisation to use? (Littelmann-Bernstein-Zelevinsky polytopes?)

Tropical geometry is a powerful tool! (and interesting in its own right..)