A tropical approach to secant varieties

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Prototypical example: polynomial interpolation

Set–up:
\(d \in \mathbb{N}\)

\(p_1, \ldots, p_k\) general points in \(\mathbb{C}^2\)

\(\text{codim}\{ f \in \mathbb{C}[x, y]_{\leq d} \mid \forall i : f(p_i) = f_x(p_i) = f_y(p_i) = 0 \} = ??\)

expect: \(\min\{3k, \binom{d+2}{2}\}\) (upper bound)

Hirschowitz (1985):
correct, unless \((d, k) = (2, 2)\) or \((d, k) = (4, 5)\) (dim 1 instead of 0)

D (2006): new proof using tropical geometry, paper and scissors

Alexander and Hirschowitz (1995): more variables
Also doable tropically??

Brannetti (2007, student of Ciliberto): three variables, tropically
Secant varieties

$C$ a closed cone in a $K$-space $V$, $k \in \mathbb{N}$

$kC := \{ v_1 + \ldots + v_k \mid v_i \in C \}$,

the $k$-th secant variety of $C$


**Example 1.**

- $C_1 = \text{rank} \leq 1$ matrices in $V_1 = M_m$
  $kC_1 = \text{rank} \leq k$ matrices

- $C_2 = \{ z_1 \wedge z_2 \} \subseteq V_2 := \wedge^2 K^m$
  cone over Grassmannian of 2-spaces in $K^m$
  $kC_2 = \text{skew-symmetric matrices of rank} \leq 2k$

- $C_3 = \text{cone over Grassmannian of isotropic 2-spaces in } K^m$
  $2C_3$ and $3C_3$ are complicated
  $kC_3 = kC_2$ for $k \geq 4$ (Baur and Draisma, 2004)
Non-defectiveness

Note: \( \dim kC \leq \min\{k \dim C, \dim V\} \), the expected dimension.

Definition 2.
\( kC \) is non-defective if \( \dim kC \) is as expected.
\( C \) is non-defective if all \( kC \) are.

Many \( C \)'s are non-defective, but hard to prove so!

Secant dimensions known for:

- Veronese embeddings (Alexander-Hirschowitz)
- certain Grassmannians and certain Segre(-Veronese) embeddings (Catalisano, Geramita, Gimigliano)
- certain highest-weight orbits (Baur, Draisma, de Graaf)
Goal of this talk

Combinatorial lower bound for $\dim kC$ where

$$C = \{ v_1^{d_1} \otimes v_2^{d_2} \otimes \ldots \otimes v_p^{d_p} \mid v_i \in \mathbb{C}^{n_i} \} \subseteq S^{d_1}(\mathbb{C}^{n_1}) \otimes \ldots \otimes S^{d_p}(\mathbb{C}^{n_p})$$

(Segre-Veronese embeddings)

**Conjecture 3.** This lower bound is always sharp.

Lots of evidence!

**Example 4.**

$$S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^2):$$

$$(\dim kC)_k = (4, 8, 12, 16, 20, 24, 26, 27)$$
Aside: relation to polynomial interpolation

\[ V = K^m \]
\[ C := \{v^d\} \subseteq S^dV \]
\[ \dim kC = ? \]

**Lemma 5** (Terracini, 1911). For \( v_1, \ldots, v_k \in V \) generic
\[ \dim kC = \dim T_{v_1}C + \ldots + \dim T_{v_k}C. \]

Lasker, 1904:
\[ T_{v_i}C = \{ f \in S^d(V^*) \mid f \text{ is singular in } [v_i] \in \mathbb{P}V \}^0 \]
so
\[ \dim kC = \text{codim} \{ \text{hom. pols. of degree } d \text{ singular in all } [v_i] \}. \]
Tropical geometry: main definition

Set-up:
\[ v : K \to \mathbb{R} := \mathbb{R} \cup \{\infty\} \text{ non-Archimedean valuation} \]

\[ (v^{-1}(\infty) = \{0\}, v(ab) = v(a) + v(b), \text{ and } v(a + b) \geq \min\{v(a), v(b)\} \]

think \( K \) =Laurent series over \( \mathbb{C} \) in \( t \)

technical conditions on \((K, v)\)

\[ X \subseteq K^n \text{ closed subvariety} \]
\[ \mapsto TX := \{v(x) = (v(x_1), \ldots, v(x_n)) | x \in X\} \]

\( \text{tropicalisation of } X \)

depends on coordinates!
Codimension one

$X$ zero set of one polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$

$\mathcal{T} f(\xi) := \min_{\alpha \in \mathbb{N}^n} (v(c_\alpha) + \langle \xi, \alpha \rangle)$ tropicalisation of $f$

**Theorem 6** (Einsiedler–Kapranov–Lind).

$\mathcal{T} X = \{ \xi \in \overline{\mathbb{R}}^n \mid \mathcal{T} f \text{ not linear at } \xi \}$

$\rightsquigarrow$ tropical hypersurfaces are polyhedral complexes!

**Example:**

$f = x_1 + x_2 - 1$ (line)

$\mathcal{T} f = \min\{\xi_1, \xi_2, 0\}$
Plane Curves: conics
Plane Curves: a cubic
Aside: counting plane curves

**Proposition 7.** \( \exists \) characterisation of tropical curves of degree \( d \) in the plane.

Mikhalkin (re)computed the number of *classical* degree \( d \), genus \( g \) plane curves through \( 3d + g - 1 \) general points:

count *tropical* such curves, each with a certain multiplicity.

Caporaso–Harris in 1998 needed starker algebraic geometry!

*algorithms* for enumerating such tropical curves
(Mikhalkin, Gathmann–Markwig)
Higher codimension

$I$ the ideal of $X \subseteq K^n$

**Theorem 8** (EKL 2004, SS 2003, see also D 2006).

\[
\mathcal{T} X = \{(v'(x_1), \ldots, v'(x_n)) \mid v' : K[X] \to \overline{\mathbb{R}} \text{ ring valuation extending } v\}
\]

\[
= \{w \in \overline{\mathbb{R}}^n \mid \forall f \in I : \mathcal{T} f \text{ not linear at } w\}
\]

**Theorem 9** (Bogart–Jensen–Speyer–Sturmfels–Thomas (2005)).

∃ finite subset of $I$ for which previous theorem is true

\[\leadsto \text{tropical basis (hard to compute!)}\]

\[\leadsto \mathcal{T} X \text{ is a polyhedral complex}\]

**Theorem 10** (Bieri–Groves (1985), Sturmfels).

\[X \text{ irreducible of dimension } d \Rightarrow \dim \mathcal{T} X = d\]
Tropical lower bounds on \( \dim kC \)

<table>
<thead>
<tr>
<th>algebraic geometry</th>
<th>tropical geometry</th>
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</thead>
<tbody>
<tr>
<td>embedded affine variety ( X \subseteq K^n )</td>
<td>polyhedral complex ( \mathcal{T}(X) \subseteq \mathbb{R}^n )</td>
</tr>
<tr>
<td>polynomial map ( f )</td>
<td>piecewise linear map ( \mathcal{T}(f) )</td>
</tr>
<tr>
<td>( \dim X )</td>
<td>( \dim \mathcal{T}(X) )</td>
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</tbody>
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Strategy: prove \( \dim \mathcal{T}(kC) = k \dim C \); then \( kC \) is non–defective. But \( kC \) not known, let alone \( \mathcal{T}(kC) \)!

Proposal:

- parameterise \( h : K^m \rightarrow C \subseteq V \)
- tropicalise \( f : (K^m)^k \rightarrow kC \), \( (z_1, \ldots, z_k) \mapsto h(z_1) + \ldots + h(z_k) \)
- compute \( \text{rk } d\mathcal{T}(f) \) at a good point \( \leadsto \) lower bound on \( \dim \mathcal{T}kC \)
A (simplified) theorem

\[ h = (h_1, \ldots, h_n) : K^m \rightarrow C \subseteq K^n \] parameterisation

assume each \( h_b = c_b x^{\alpha_b} \neq 0 \) (1 term)

for \( l = (l_1, \ldots, l_k) \) \( k \) affine–linear functions on \( \mathbb{R}^m \) set
\[ C_i(l) := \{ \alpha_b | l_i(\alpha_b) < l_j(\alpha_b) \text{ for all } j \neq i \} \]

**Theorem II** (Draisma, 2006).

\[
\dim kC \geq \sum_i (1 + \dim \text{Aff}_{\mathbb{R}} C_i(l))
\]

Lower bound=3+2+1
Funny optimisation problem

\[ A \subseteq \mathbb{R}^n \text{ finite, } k \in \mathbb{N} \]

Maximise \( \sum_i (1 + \dim \text{Aff}_\mathbb{R} C_i(l)) =:* \)

over all \( l = (l_1, \ldots, l_k) \), each \( l_i \) affine–linear

**Corollary 12.**
\[ A = \{ \alpha_b \mid b \} \] exponents of monomials in parameterisation
draw \( A \) on \( m \)–dimensional paper
cut paper into \( k \) pieces
compute sg. like *
\( \leadsto \) lower bound on \( \dim kC \)
Generalisation of Theorem 11

Optimisation problem:
Given
\( k \in \mathbb{N} \)
\( A = (A_1, \ldots, A_n) \) list of finite subsets of \( \mathbb{R}^n \)

Optimisation domain
\( k \)-tuples \( l = (l_1, \ldots, l_k) \) of affine linear functions on \( \mathbb{R}^n \)

Objective function
\( \sum_{i=1}^{k} (1 + \dim \text{Aff}_{\mathbb{R}} C_i(l)) \)
where \( C_i = \bigcup_{b=1}^{n} \{ \alpha \in A_b \mid f_i(\alpha) < f_j(\beta) \text{ for all } (\beta, j) \in A_b \times \{1, \ldots, k\} \} \)

Optimal value \( \text{AP}^*(A, k) \)

Theorem 13. \( h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n \) parametrisation
\( A_b \subseteq \mathbb{N}^m \) support of \( h_b \)
then \( \text{AP}^*(A, k) \leq \dim kC \)
Results with Karin Baur

Non–degenerate:

• all Segre–Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 2)$ (Catalisano-Geramita-Gimigliano)

• all Segre–Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 1, 1)$ (Catalisano-Geramita-Gimigliano)

• Segre embedding of $(\mathbb{P}^1)^6$ (cells for $k = 9$: 8 disjoint Hamming balls of radius 1 and one cell in the middle)

• $\{\text{flags point} \subset \text{line} \subset \mathbb{P}^2\}$ non–defective in all embeddings except those of highest weight $\omega_1 + \omega_2$ (adjoint representation) or $2\omega_1 + 2\omega_2$

• all Segre–Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^2$ except $(2, \text{even})$ and $(3, 1)$
Rationale

Results by others:

- Veronese embeddings of $\mathbb{P}^3$ (Brannetti)
- Segre embeddings of $(\mathbb{P}^1)^d$ for $d = 1, \ldots, 9$
  (Halupczok, tropically with computer)

**Conjecture:** tropical lower bound sharp for all Segre-Veronese embeddings.
Some pictures

picture suggests: S-V embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2, 2, 2)$ has defective $7C$. Indeed!

Minimal orbit in representation of $SL_3$ of highest weight $(5, 1)$ is non-defective.
Conclusion

Non-defectiveness often provable by optimising a strange polyhedral-combinatoric objective function.

Hope a point in $T(kC)$ with full-dimensional neighbourhood gives restrictions on the ideal of $kC$. Sufficient to settle one or two more cases of GSS?

Segre-Veronese is the given bound always correct?

Other minimal orbits Smallest flag variety doable with a trick: reduce all $A_b$ in Theorem 13 to singletons, and use Voronoi-variant. In general: which parametrisation to use? (Littelmann-Bernstein-Zelevinsky polytopes?)