Tropical reparameterisations

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Two ways to describe a line

implicitly, by equations
\[ X := \{(x, y) \mid y - x - 1 = 0\} \subset \mathbb{A}^2 \]

explicitly, by parameterisation
\[ \phi : \mathbb{A}^1 \to \mathbb{A}^2, \quad u \mapsto (u, u + 1); \quad X = \text{im } \phi \]

**elimination theory:** parameterisation \(\sim\) equations?
Tropicalising those two ways

by equations
\[ X = \{(x, y) \mid y - x - 1 = 0\} \subset \mathbb{A}^2 \]
\[ TX = \{ (\xi, \eta) \mid \min\{\eta, \xi, 0\} \text{ is attained at least twice} \} \subset \mathbb{R}_\infty^2 \]

by parameterisation
\[ \phi : u \mapsto (u, u + 1) \]
\[ \mathcal{T}\phi : v \mapsto (v, \min\{v, 0\}) \]

\[ \text{im} \mathcal{T}\phi \subseteq TX, \text{ in general} \subseteq \]
Reparameterisation for the line

\[ \alpha : \mathbb{A}^1 \to \mathbb{A}^1, \quad s \mapsto s - 1 \]
\[ \phi' := \phi \circ \alpha : \mathbb{A}^1 \to \mathbb{A}^2, \quad s \mapsto (s - 1, s) \]
\[ \mathcal{T}(\phi') : \mathbb{R}_\infty \to \mathbb{R}_\infty^2, \quad \sigma \mapsto (\min\{\sigma, 0\}, \sigma) \]

Two reparameterisations tropically cover \( \mathcal{T}X \)
Four questions

\( \phi : \mathbb{A}^m \to \mathbb{A}^n \) polynomial map
\( X := \text{im} \phi \text{ algebraic variety} \)
then \( \text{im} \mathcal{T} \phi \subseteq \mathcal{T} X \)

\[ \exists \, ? \text{ finitely many (or one) reparameterisations } \alpha_i: \mathbb{A}^{p_i} \to \mathbb{A}^m \]
(or rational maps) such that \( \bigcup_i \text{im} \mathcal{T} (\phi \circ \alpha_i) = \mathcal{T} (X) \).

**Remark.** Sturmfels-Tevelev-Yu (2007) describe \( \mathcal{T} X \) from \( \phi \) in case of generic coefficients; generalisations use Hacking-Keel-Tevelev’s geometric tropicalisation (2007).
Two observations

**Lemma.** If $\phi = (\phi_1, \ldots, \phi_n)$ with all $\phi_i$ homogeneous of same degree, then the four questions are equivalent.

Multiply with common denominator; combine several reparameterisations into one.

**Proposition.** All four questions reduce to the case where $X$ is a hypersurface in $\mathbb{A}^n$.

Choose “generic” monomial map $\pi : \mathbb{A}^n \to \mathbb{A}^{d+1}$ where $d = \dim X$; reparameterisations that work for $\pi \circ \phi$ also work for $\phi$. 
Linear spaces

**Theorem (Yu-Yuster, 2006).** \( \phi : \mathbb{A}^m \to X \subseteq \mathbb{A}^n \) linear, given by a matrix \( \phi \)

Then \( \text{im} \, T \phi = T \, X \iff \) every vector \( v \in X \) of minimal support (cocircuit) is scalar multiple of a column of \( \phi \).

(This can be achieved form by composing \( \phi \) with a linear map \( \mathbb{A}^p \to \mathbb{A}^m \).)

**Example.** \( \phi : \mathbb{A}^2 \to \mathbb{A}^3 \) given by \( \phi = \begin{bmatrix} t & 0 \\ 0 & 1 \\ 1 & t \end{bmatrix} \) over \( \mathbb{C}((t)) \)

\( X = \{ (x, y, z)^T \mid x + t^2 y - tz = 0 \} \)

\( T \, X = C_1 \cup C_2 \cup C_3 \) with

\( C_1 = \{ (\xi, \xi - 2, \zeta) \mid \zeta \geq \xi - 1 \} \)

\( C_2 = \{ (\xi, \eta, \xi - 1) \mid \eta \geq \xi - 2 \} \)

\( C_3 = \{ (\xi, \eta, \eta + 1) \mid \xi \geq \eta + 2 \} \)

\( T \phi : (\alpha, \beta) \mapsto (\alpha + 1, \beta, \min\{\alpha, \beta + 1\}) \); \( \text{im} \, T \phi = C_2 \cup C_3 \)
Example, continued

\[ \phi \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} t & 0 & t^2 \\ 0 & 1 & -1 \\ 1 & t & 0 \end{bmatrix} \]

The last matrix contains all cocircuits of \( X \), so

\[ \text{im} \mathcal{T} \left( \phi \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} \right) = C_1 \cup C_2 \cup C_3 = \mathcal{T} X \]

by Yu and Yuster’s theorem.
A Grassmannian from a linear space

\[ \phi : \mathbb{A}^n \to X \subseteq \mathbb{A}^{\binom{n}{2}}, \quad (x_1, \ldots, x_n) \mapsto (x_i - x_j)_{i < j} \]

zero patterns in the image \( \rightsquigarrow \) partitions of \( \{1, \ldots, n\} \)
cocircuits \( \rightsquigarrow \) partitions into two parts
so \( \exists \alpha : \mathbb{A}^{2^{n-1}-1} \to \mathbb{A}^n \) linear with \( \text{im} \, \mathcal{T}(\phi \circ \alpha) = \mathcal{T}X \)

\[ \psi : \mathbb{A}^n \times \mathbb{A}^n \to Y \subseteq \mathbb{A}^{\binom{n}{2}}, \quad (u, x) \mapsto (u_i u_j (x_i - x_j))_{i < j} \]

\( Y = \text{Grassmannian of 2-dimensional subspaces of } n\text{-space} \)

\( \text{im} \, \mathcal{T}(\psi \circ (\text{id} \times \alpha)) = \mathcal{T}Y \), the \textit{tropical Grassmannian} studied by Speyer and Sturmfels (2004) and many others

Points of \( Y \) correspond to \textit{tree metrics}, by the above obtained from tropical linear combinations of 2-partitions by stretching ends.
Example with $n = 4$

$\{1, 234\}$ short-hand for $(d_{ij})_{i<j}$ with $d_{1j} = 0$ and all other $d_{ij} = \infty$

$$(1 \otimes \{1, 234\}) \oplus (2 \otimes \{13, 24\}) \oplus (3 \otimes \{14, 23\})$$

equals the tree metric of

![Tree Metric Diagram]

**Remark.** Internal edges have negative length. Edges adjacent to leaves can be arbitrarily altered using the $\nu_i$. 
Local tropical reparameterisations

\[ \phi : \mathbb{A}^m \to \mathbb{A}^n \text{ polynomial map} \]

\[ X := \text{im } \phi \text{ algebraic variety of dimension } d \]

**Theorem.** For almost all \( \xi \in TX \)

\[ \exists \alpha : T^d \to \mathbb{A}^m \text{ such that} \]

\[ \text{im } T (\phi \circ \alpha) \supset a d\text{-dimensional neighbourhood of } \xi. \]

**Remark.** • \( \alpha \) is allowed to have Laurent polynomial components

• \( d \) is also the dimension of \( TX \)

• if all \( \phi_i \) homogeneous of the same degree, \( k \) such local reparameterisations can be combined to a reparameterisation \( \mathbb{A}^{kd} \to \mathbb{A}^m \)

• *almost all* means the \( \xi_i \) span a \( d \)-dimensional \( \mathbb{Q} \)-subspace of \( \mathbb{R} \)
Proof sketch

1. assume $\xi_1, \ldots, \xi_d$ linearly independent over $\mathbb{Q}$
2. consider $K = \mathbb{C}(t_1, \ldots, t_d)$ with valuation $v(t_i) = \xi_i$
3. take a point $p$ of $\mathbb{A}^m$ with coordinates in $\overline{K}$ such that $v(\phi(p)) = \xi$; exists
4. approximate $p$ with a point $q$ in $\mathbb{C}[t_1^{\pm1/N}, \ldots, t_d^{\pm1/N}]$ such that $v(\phi(q)) = \xi$ (multivariate Puiseux theorem)
5. set $u_i := t_i^{1/N}$
6. $q(u_1, \ldots, u_n)$ is the required reparameterisation

Remark. • not yet very constructive, but I’m collaborating with Anders Jenssen to make it so
• not clear that finitely many suffice...