Tropical reparameterisations

Jan Draisma
j.draisma@tue.nl

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SIAM conference
Applications of Algebraic Geometry
North Carolina State University
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Reparameterising a line

\( \varphi : (\mathbb{C}^*) \rightarrow (\mathbb{C}^*)^2, t \mapsto (t + 1, t) \)

\( \text{Trop}(\varphi) : \mathbb{R} \rightarrow \mathbb{R}^2, \\ t \mapsto (\min\{t, 0\}, t) \)

Not surjective!

Coordinate change

\( \alpha : \mathbb{C}^* \rightarrow \mathbb{C}^*, s \mapsto s - 1 \)

\( \varphi \circ \alpha : s \mapsto (s, s - 1) \)

\( \text{Trop}(\varphi \circ \alpha) : s \mapsto (s, \min\{s, 0\}) \)
Problem statement

\((K, v)\) non-Archimedean field
\(\varphi : (K^*)^m \rightarrow (K^*)^n\) rational map
\(X := \text{im } \varphi\) unirational
\(\text{Trop}(\varphi) : \mathbb{R}^m \rightarrow \mathbb{R}^n\)
(replace \(c \in K\) by \(v(c)\),
+ by \(\min\), times by addition,
division by subtraction)

Fact
\(\text{Trop}(\varphi)\) maps \(\mathbb{R}^m\) into \(\text{Trop}(X)\).

Question
\(\exists p \in \mathbb{N}\) and \(\alpha : (K^*)^p \rightarrow (K^*)^m\)
such that \(\text{Trop}(\varphi \circ \alpha)\) surjective
onto \(\text{Trop}(X)\)?

If yes, call \(\varphi\) \textit{tropically reparameterisable}.

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\(\exists p \in \mathbb{N}\) and \(\alpha : (K^*)^p \rightarrow (K^*)^m\)
such that \(\text{Trop}(\varphi \circ \alpha)\) surjective onto \(\text{Trop}(X)\)?

If yes, call \(\varphi\) tropically reparameterisable.

Known

Lemma
- Question is equivalent to:
  \(\exists (p_i, \alpha_i), i = 1, \ldots, k\) such that
  \(\bigcup_i \text{Trop}(\varphi \circ \alpha_i) = \text{Trop}(X)\)?
- W.l.o.g. \(X\) is a hypersurface.
- W.l.o.g. \(\varphi\) is birational to \(X\).
  (But perhaps not w.l.o.g. both.)
- Need \(K\) algebraically closed.

Tropically reparameterisable
- \(m = 1\): curves (Speyer)
- \(\varphi\) linear (Yu-Yuster)
- if \(\varphi\) is, then so is \(\mu \circ \varphi\) with \(\mu\) monomial
- \(\text{Gr}(2, n)\), rank-2 matrices, …
- Horn uniformisation of
  \(A\)-discriminants a (Dickenstein-Feichtner-Sturmfels)
Singular matrices

\[ \varphi : (K^*)^{4 \times 3} \times (K^*)^{3 \times 4} \rightarrow (K^*)^{4 \times 4}, \]

\[(A, B) \mapsto AB \]

\[ X = V(\det) \]

Maximal cones of \( \text{Trop}(X) \):

\[ \frac{4^2 3! 2!}{2} \]

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\[ (4^2) \cdot 2! \cdot \frac{2! 1!}{2} \]

Observation

For \((i, j) \in [4]^2\) set \( Y_{ij} := \) union of cones with full-dimensional projection \( \text{pr} \) along \((i, j)\)-entry.

\[ \leadsto \text{pr} : Y_{ij} \rightarrow \mathbb{R}^{[4]^2-(i,j)} \text{ bijective!} \]

\( \leadsto 4^2 \) reparameterisations like

\[ (K^*)^{3 \times 3} \times ((K^*)^{3})^2 \rightarrow (K^*)^{4 \times 4}, \]

\[(A, v, w) \mapsto \begin{bmatrix} A & v \\ w^t & w^t A^{-1}v \end{bmatrix}; \]

tropicalisations cover \( \text{Trop}(X) \).

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Lemma

- Question is equivalent to:

\[ \exists (p_i, \alpha_i), i = 1, \ldots, k \text{ such that } \bigcup_i \text{Trop}(\varphi \circ \alpha_i) = \text{Trop}(X)? \]

- W.l.o.g. \( X \) is a hypersurface.

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Singular matrices
\[ \varphi : (K^*)^{4 \times 3} \times (K^*)^{3 \times 4} \to (K^*)^{4 \times 4}, \]
\[ (A, B) \mapsto AB \]
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Maximal cones of \( \text{Trop}(X) \):
\[ \binom{4}{0} \frac{4!3!}{2} \quad \binom{4}{1} \frac{3!2!}{2} \quad \binom{4}{2} \frac{2!}{2} \cdot 2! \cdot \frac{2!1!}{2} \]

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\[ \ni \text{pr} : Y_{ij} \to \mathbb{R}^{[4]^2 - (i, j)} \text{ bijective!} \]
\[ \ni 4^2 \text{ reparameterisations like} \]
\[ (K^*)^{3 \times 3} \times ((K^*)^3)^2 \to (K^*)^{4 \times 4}, \]
\[ (A, v, w) \mapsto \begin{bmatrix} A & v \\ w^t & w^t A^{-1}v \end{bmatrix}; \]
tropicalisations cover \( \text{Trop}(X) \).

Theorem
\( \text{Trop}(V(\det)) \) is \textit{tropically unirational}: the image of \( \mathbb{R}^{n^2 \times (n^2 - 1)} \) under some tropical rational map.

A local result
\( \text{char } K = 0 \) and \( \dim_{\mathbb{Q}} \psi(K^*) \text{ finite} \)

Theorem
\( y = (y_1, \ldots, y_n) \in \text{Trop}(X) \)
\text{very generic on a dimension-}d \text{ polyhedron } P \text{ of } \text{Trop}(X) \text{ Then } \exists \alpha : (K^*)^d \to (K^*)^m \text{ s.t.} \]
\( \text{Trop}(\varphi \circ \alpha) \) hits an open neighbourhood of \( y \) in \( P \).

Very generic means \( \langle y_1, \ldots, y_n \rangle_{\mathbb{Q}} \mod \psi(K^*) \text{ has } \mathbb{Q}\text{-dimension } d. \)
Theorem
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Very generic means \(\langle y_1, \ldots, y_n \rangle_{\mathbb{Q}} \mod v(K^*)\) has \(\mathbb{Q}\)-dimension \(d\).