Well-quasi-orders and algebra

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Well-quasi-orders

**Definition.** A well-quasi-order on a set $A$ is a transitive relation $\preceq$ s.t. for all $a_1, a_2, \ldots$ there exist $i < j$ with $a_i \preceq a_j$.

(Non-)examples

- Any transitive relation on a finite set $A$ is a wqo.
- The standard $\leq$ on $\mathbb{N} = \{1, 2, \ldots\}$ is a wqo; $|$ is not.
- The standard $\leq$ on $\mathbb{Z}$ is not a wqo: $-1, -2, \ldots$.
- $A$ is infinite $\Rightarrow =$ is not a wqo: infinite antichains.

**Lemma.** If $\preceq$ is a wqo on $A$ and $a_1, a_2, \ldots \in A$, then \exists i_1 < i_2 < \ldots$ with $a_{i_1} \preceq a_{i_2} \preceq \ldots$

**Dickson’s Lemma.** $\preceq_i$ a wqo on $A_i$ for $i = 1, 2 \leadsto (a, b) \preceq (c, d) :\Leftrightarrow (a \preceq_1 c \text{ and } b \preceq_2 d)$ is a wqo on $A_1 \times A_2$. 
Higman’s lemma

\[ A^* = \bigcup_{d=0}^{\infty} A^d \] is the set of words on \( A \)

Lemma (Higman)
If \( \preceq \) is a wqo on \( A \), then \( \preceq \) on \( A^* \) defined by \( (a_1, \ldots, a_d) \preceq (b_1, \ldots, b_e) :\iff \exists \) strictly increasing \( \pi : [d] \to [e] \) with \( \forall i : a_i \preceq b_{\pi(i)} \) is a wqo.

Proof
• If not, take a bad sequence, i.e., \( w_1, w_2, \ldots \in A^* \) with \( \nexists i < j : w_i \preceq w_j \); minimal in the following sense: \( w_k \) is shortest among all bad sequences starting with \( w_1, \ldots, w_{k-1} \).
• Write \( w_i = (a_i, u_i) \) and find \( i_1 < i_2 < \ldots \) with \( a_{i_1} \preceq a_{i_2} \preceq \ldots \)
• Now \( w_1, \ldots, w_{i_1-1}, u_{i_1}, u_{i_2}, \ldots \) is a bad sequence \( \rightsquigarrow \) contradiction.
\[ \Box \]
Further well-quasi-orders

**Theorem (Kruskal)**
Generalisation of Higman’s lemma to rooted, $A$-labelled trees.

**Theorem (Maclagan)**
The set of monomial ideals in $K[x_1, \ldots, x_n]$ is well-quasi-ordered by $I \preceq J := I \supseteq J$. 
A new well-quasi-order

Tensor restriction theorem (Blatter-D-Rupniewski)
\( \mathbb{F}_q \) a finite field, \( d \in \mathbb{Z}_{\geq 0} \), \( V_1, V_2, \ldots \) f.d. \( \mathbb{F}_q \)-spaces, \( T_i \in V_i \otimes^d_i \). Then \( \exists i < j, \phi \in \text{Hom}(V_j, V_i) : \phi \otimes^d T_j = T_i \).

Corollary
Every property of order-\( d \) tensors over \( \mathbb{F}_q \) preserved under applying linear maps can be tested in polynomial time.

Multilinear analogue of:

Theorem (Robertson-Seymour)
The minor order on finite, undirected graphs is a wqo.
Prototypical use of a wqo in algebra

\( K \) a field (or Noetherian ring), fixed throughout

**Hilbert’s basis theorem**
Every ideal \( I \) in \( K[x_1, \ldots, x_n] \) is finitely generated.

**Proof template:**
- monomials in \( x_1, \ldots, x_n \) are wqo wrt \( x^\alpha | x^\beta \) (Dickson);
- hence with respect to any monomial order \( \leq \) the set \( \text{lm}(I) = \{\text{lm}(f) \mid f \in I \setminus \{0\}\} \) has finitely many \( \mid \)-minimal elements: \( \text{lm}(f_1), \ldots, \text{lm}(f_k) \).
- \( f_1, \ldots, f_k \) generate \( I \) (division with remainder). \( \square \)
Linear algebra over categories
**Definition.** $C$ a category $\rightsquigarrow$ a $C$-module over $K$ is a (covariant) functor $M : C \rightarrow \text{Mod}_K$, i.e.:

- $\forall S : M(S)$ is an $K$-module;
- $\forall \pi \in \text{Hom}_C(S, T) : M(f) : M(S) \rightarrow M(T)$ is $K$-linear;
- and $M(1_S) = 1_{M(S)}$ and $M(\sigma \circ \pi) = M(\sigma) \circ M(\pi)$.

**Remarks**

- $C$-modules over $K$ form an abelian category.
- Many natural notions, such as finitely generated.
- Each $M(S)$ is a representation of $\text{Aut}_C(S)$.

**Example**

$C = \text{FI}$: Finite sets with Injections

$M(S) = K \cdot \binom{S}{2}$, generated by $1 \cdot \{1, 2\} \in M([2])$
**FI-modules in the wild**

**Example**

$X$ a manifold $\rightsquigarrow \text{Conf}_X : \text{FI}^{\text{op}} \to \text{manifolds}$, defined by $\text{Conf}_X(S) = \{\text{injective maps } S \to X\}$; the *pure configuration space* of $X$.

**Theorem (Church-Ellenberg-Farb)**

Fix $p \geq 0$. Under mild conditions on $X$, $H^p(\text{Conf}_X, \mathbb{Q})$ is a finitely generated $\text{FI}$-module over $\mathbb{Q}$.

Nice consequences, e.g. $\dim_{\mathbb{Q}}(H^p(\text{Conf}_X([n]), \mathbb{Q}))$ is polynomial in $n$ for $n \gg 0$, and splits into a fixed number of $S_n$-representations (*representation stability*).
Theorem (Church-Ellenberg-Farb). Every sub-$\text{FI}$-module $V$ of a finitely generated $\text{FI}$-module $M$ is finitely generated.

Same proof template (Sam-Snowden)

- Work with $\text{OI}$: sets $[d]$, $d \in \mathbb{Z}_{\geq 0}$ with increasing maps.
- $M$ is a quotient of $P = P_{d_1} \oplus \cdots \oplus P_{d_k}$, where $P_d(S) = K \cdot \text{Hom}_{\text{OI}}([d], S)$; suffices to prove for $M = P$.

- For basis elements $\pi \in \text{Hom}_{\text{OI}}([d], S) \subseteq P_d(S)$ and $\sigma \in \text{Hom}_{\text{OI}}([d], T)$, write $\pi \preceq \sigma$ if $\exists \varphi \in \text{Hom}_{\text{OI}}(S, T) : \sigma = \varphi \circ \pi$.
- This is a wqo on the basis in each $P_d$ (Higman’s lemma for $A = \{0, 1\}$ with $=\$), hence on the basis in $P$.

- Choose an $\text{OI}$-compatible linear order on the basis in each $P(S)$. Then $\exists$ finitely many $v_i \in V(S_i)$ s.t. $\forall S$, $\forall v \in V(S) \exists i : \text{Im}(v_i) \preceq \text{Im}(v)$. These generate $V \subseteq P$. □
Similar finiteness results

Sam-Snowden call $\text{OI}$ a \textit{Gröbner category}, and $\text{FI}$ a \textit{quasi-Gröbner category}.

$\text{FS}$: Finite sets and Surjective maps  
$\text{FS}^{\text{op}}$: the opposite category

\textbf{Theorem (Sam-Snowden)} \newline \text{FS}^{\text{op}} \text{ is quasi-Gröbner. Hence any sub-}\text{FS}^{\text{op}}\text{-module of a finitely generated } \text{FS}^{\text{op}}\text{-module is finitely generated.}

\textbf{Proof}: (not so easy) exercise: find an ordered version of $\text{FS}^{\text{op}}$ and a suitable wqo, etc.

$\text{FI}$-modules appear naturally throughout math. \newline $\text{FS}^{\text{op}}$-modules not so much, but . . .
The Lannes-Schwartz Artinian conjecture

Let $\mathbb{F}_q$ be a finite field, $\text{Vec}_{\mathbb{F}_q}$ the category of finite-dimensional vector spaces over $\mathbb{F}_q$.

Corollary (Putman, Sam, Snowden)
Finitely generated $\text{Vec}_{\mathbb{F}_q}$-modules $M$ over $K$ are Noetherian.

Proof

• Have a functor $Q : \text{FS}^\text{op} \to \text{Vec}_{\mathbb{F}_q}$, $Q(S) = \mathbb{F}_q S$.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

• Show that $M \circ Q$ is a finitely generated $\text{FS}^\text{op}$-module: every $T \times$ bounded matrix over $\mathbb{F}_q$ is of the form $Q(\pi) \cdot A$ for some bounded $\times$ bounded matrix $A$. □
Commutative algebra over categories
Algebras over a category

**Definition.** A \( C \)-algebra over \( K \) is a functor from \( C \) to (commutative, unital) \( K \)-algebras.

Natural notions of ideals and Noetherianity.

Coordinate rings of matrix spaces as FI-algebras

- \( B(S) := K[x_{ij} \mid i, j \in S] \)
- \( A_c(S) := K[x_{ij} \mid i \in [c], j \in S] \)

Examples

- The ideals \( I_k \subseteq A \) and \( J_k \subseteq B \) generated by all \( k \times k \)-determinants is finitely generated in both \( B \) (by \( k + 1 \) elements) and in \( A \) (by \( \binom{c}{k} \) elements).
- \( B \) is not Noetherian.
Cohen’s theorem and applications

**Theorem (Daniel Cohen, 1987)**
The FI-algebra $A_c : S \mapsto K[x_{ij} | i \in [c], j \in S]$ is Noetherian.

Many, many applications and follow-up work:
- the independent set theorem (Hillar-Sullivant)
- biv. Hilbert series (Nagel-Römer, Krone-Leykin-Snowden)
- co-dimension, projective dimension, regularity (Van Le-Nagel-Nguyen-Römer)
- moment varieties of mixtures of products (Alexandr-Kileel-Sturmfels, . . .)

**Theorem (D-Eggermont-Farooq-Meier)**
For any homomorphism $R \to A_c$ of finitely generated FI-algebras, the image closure of $\text{Spec}(A_c) \to \text{Spec}(R)$ is set-theoretically defined by finitely many equations.
Fix $n$.

For any finite set $S$, define $V(S) := (K^n)^{\otimes S}$, and for any surjective $\pi : T \to S$ define $V(\pi) : V(T) \to V(S)$ by $\bigotimes_{j \in T} v_j \mapsto \bigotimes_{i \in S} \otimes_{j \in \pi^{-1}(i)} v_j$, where $\otimes =$. 

The locus $X(S) \subseteq V(S)$ of rank-1 tensors is an $\text{FS}$-variety.

**Theorem (D-Oosterhof)**

Coordinate ring $S \hookrightarrow K[X(S)]$ is a Noetherian $\text{FS}^{\text{op}}$-algebra. (Proof uses Maclagan’s theorem.)

$\hookrightarrow$ Ideals of *iterated toric fibre products* of undirected discrete graphical models stabilise as the number of factors tend to $\infty$ (builds on work by Rauh-Sullivant and Kahle-Rauh).
The tensor restriction theorem

Theorem (Blatter-D-Rupniewski)
Let \( P : \text{Vec}_{\mathbb{F}_q} \rightarrow \text{Vec}_{\mathbb{F}_q} \) be any functor of finite length. Then the \( \text{Vec}_{\mathbb{F}_q}^{\text{op}} \)-algebra \( V \mapsto \mathbb{F}_q^{P(V)} = SP(V)^* / \langle f^q - f | f \rangle \) is Noetherian.

Corollary
Given \( p_i \in P(V_i), i = 1, 2, \ldots, \) there exist \( i < j \) and \( \varphi : V_j \rightarrow V_i \) with \( P(\varphi)p_j = p_i \). (Notation: \( p_i \preceq p_j \))

Proof
Let \( I_i \) be the ideal of functions that vanish on all \( p \) with \( p_j \not\preceq p \) for all \( j = 1, \ldots, i \). Then \( I_{j-1} = I_j \) for some \( j \) and hence \( p_i \preceq p_j \) for some \( i < j \).

Thank you!