Dualities in Logic

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PHDS IN LOGIC XI
EXAKTE WISSENSCHAFTEN
UNIVERSITY OF BERN

- Part I: Between syntax and semantics
    - Logical validity and truth
    - Stone duality for Boolean algebras
    - Some examples
- Part II: Quantifiers and definability
    - Logic on words
    - Uniform interpolation for IPC

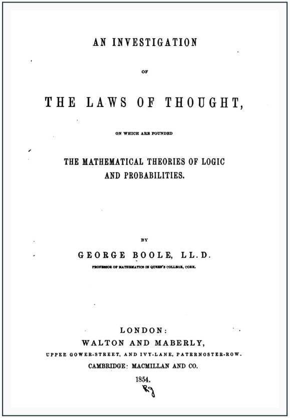
Part I

Between syntax and semantics

A gentle introduction to duality

Logical validity and truth

- Boole: An Investigation of the Laws of Thought, The Mathematical Theories of Logic and Probabilities (1854)



AN INVESTIGATION

OF

THE LAWS OF THOUGHT,

ON WHICH ARE FOUNDED

THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.

BY

GEORGE BOOLE, LL.D.

PROFESSOR OF MATHEMATICS IN QUEEN'S COLLEGE, CORK.

LONDON:

WALTON AND MABERLY,

UPPER GOWER-STREET, AND IVY-LANE, PATERNOSTER-ROW.

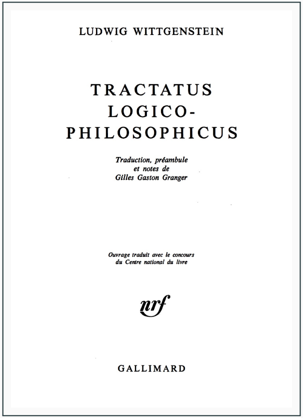
CAMBRIDGE: MACMILLAN AND CO.

1854.

Algebraization of logic

Logical validity and truth

- Boole: An Investigation of the Laws of Thought, The Mathematical Theories of Logic and Probabilities (1854)
- Wittgenstein: Tractatus Logico-Philosophicus (1922)


LUDWIG WITTGENSTEIN

TRACTATUS
LOGICO-
PHILOSOPHICUS

Traduction, préambule
et notes de
Gilles Gaston Granger

Ouvrage traduit avec le concours
du Centre national du livre

nrf

GALLIMARD

Distinction logical validity / truth

5.14 – Si une proposition suit d'une autre, celle-ci dit plus que celle-là, celle-là moins que celle-ci.

5.141 Si p suit de q et q suit de p, p et q ne sont qu'une seule et même proposition.

5.142 La tautologie suit de toute proposition : elle ne dit rien.

$$P = \{p \mid p \text{ is a proposition}\}$$

1) $p \vdash q$ iff $p \leq q$; 2) $p \vdash q$ and $q \vdash p$ iff $p \equiv q$; 3) $\bot \leq p \leq \top$.

5.123 – Si un dieu crée un monde dans lequel certaines propositions sont vraies, il crée du même coup un monde dans lequel sont valables toutes leurs conséquences. Et de même il ne saurait créer aucun monde où serait vraie la proposition « p » sans créer en même temps tous les objets de celle-ci.

$p \vdash q$ implies $p \vDash q$

5.153 – Une proposition n'est, en elle-même, ni probable ni improbable. Un événement se produit ou ne se produit pas, il n'y a pas de milieu.

For every model (=possible world) M,

$$h_M \colon P \to \{\bot, \top\}$$

is given by $h_M(p) = \top$ if, and only if, $M \vDash p$.

- $\mathbf{L}(T)$: **Lindenbaum-Tarski algebra** of a propositional theory T in a finite propositional language

$$\overline{p} = \{p_1, \ldots, p_n\}.$$

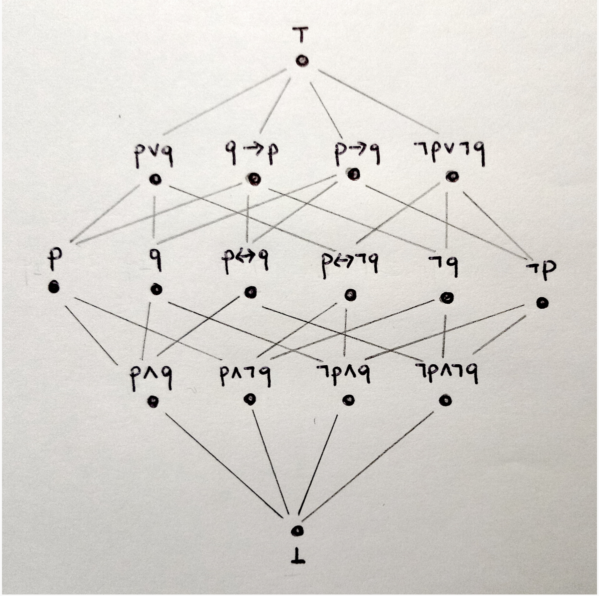
- As a set, $\mathbf{L}(T)$ consists of equivalence classes $[\varphi]$ of formulas modulo T-equiprovability, i.e.

$$[\varphi] = [\psi] \;\Leftrightarrow\; T \vdash \varphi \to \psi \text{ and } T \vdash \psi \to \varphi.$$

- $\mathbf{L}(T)$ carries a natural structure of **Boolean algebra** (BA), where:

$$0 = [\bot],\; 1 = [\top],\; [\varphi] \vee [\psi] = [\varphi \vee \psi],\; \neg[\varphi] = [\neg\varphi].$$

- If $T = \emptyset$, $\mathbf{L}(T) = \mathbf{F}(\overline{p})$ is the **free** Boolean algebra on the set $\overline{p}$.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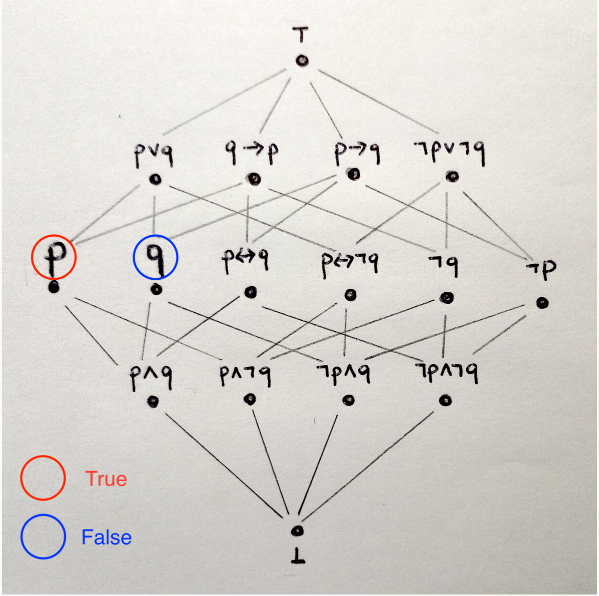
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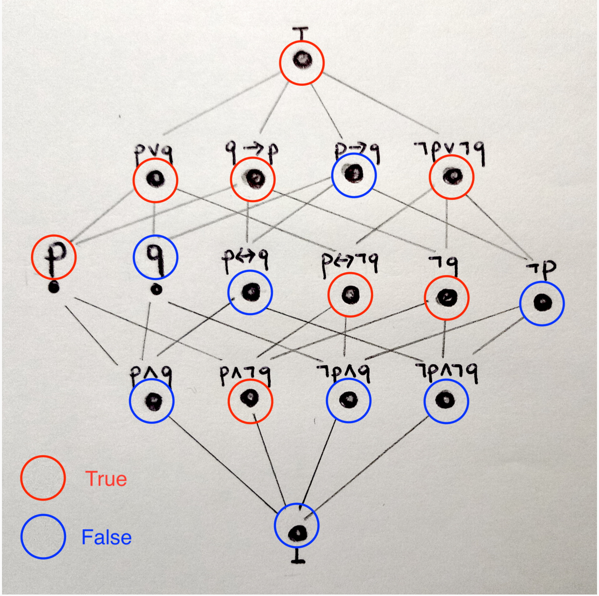
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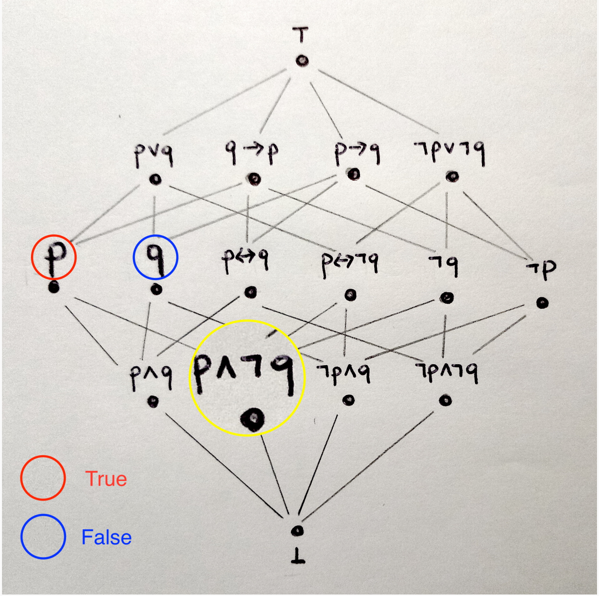
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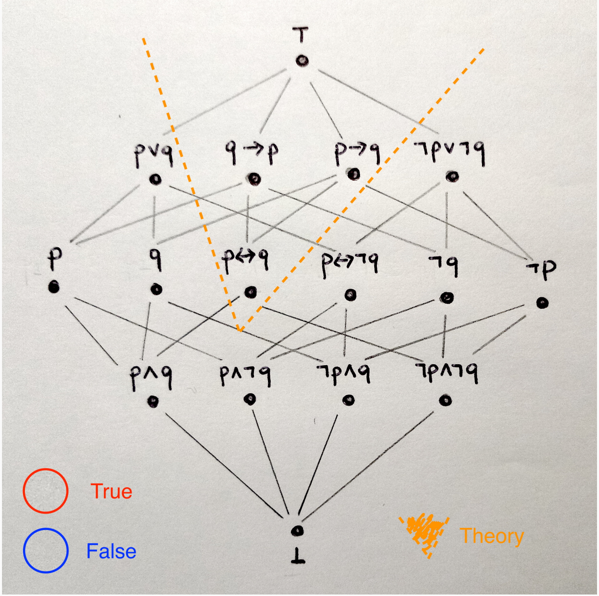
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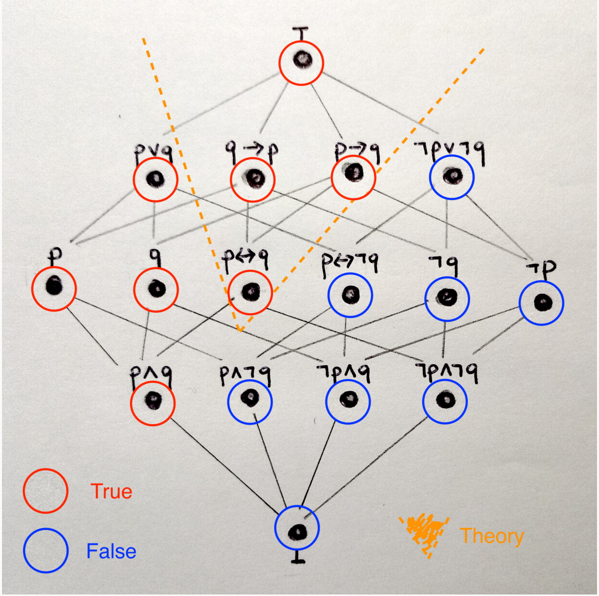
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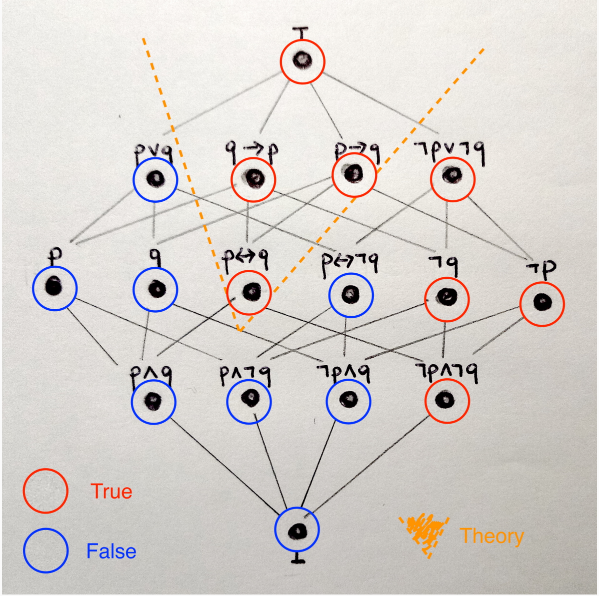

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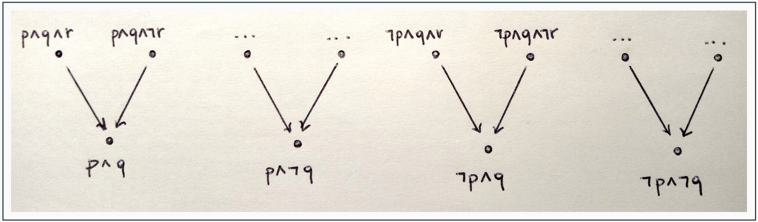
Theory

- **More** information means **less** possible worlds: the quotient $\mathbf{F}(3) \twoheadrightarrow \mathbf{L}(T)$ corresponds to an inclusion $\mathbf{At}(\mathbf{L}(T)) \hookrightarrow \mathbf{At}(\mathbf{F}(3))$.
- Similarly, **less** information means **more** possible worlds. For example, the inclusion of subalgebras

$$\mathbf{F}(2) \hookrightarrow \mathbf{F}(3)$$

corresponds to the "cylindrification"

$$\mathbf{At}(\mathbf{F}(2)) \twoheadleftarrow \mathbf{At}(\mathbf{F}(3)) \cong \mathbf{At}(\mathbf{F}(2)) \times \{0,1\}$$



Stone duality for Boolean algebras

- Given any finite Boolean algebra B, consider its set of atoms $\mathbf{At}(B)$.
- Every finite BA is the Lindenbaum-Tarski algebra of a theory T in finitely many propositional variables. So we can still regard $\mathbf{At}(B)$ as a set of **possible worlds**.
- The Boolean algebra B is recovered, up to iso, as

$$B \cong \wp(\mathbf{At}(B)),\ b \mapsto \{x \in \mathbf{At}(B) \mid x \leq b\}.$$

- In other words, **a formula can be identified with the set of all worlds in which it is satisfied**.
- Since any two atoms are incomparable, $\mathbf{At}(B)$ carries no order. It is simply a finite set. In fact, every finite set S is of this form: $S \cong \mathbf{At}(\wp(S))$.

The duality is not only at the level of objects. It also takes into account morphisms:

Lemma (exercise!)
Let S, T be finite sets. For every Boolean algebra homomorphism $h\colon \wp(T) \to \wp(S)$ there is a unique function $f: S \to T$ s.t. $h = f^{-1}$.

We obtain **finite Stone duality for Boolean algebras**:

- every finite BA is isomorphic to the power-set algebra of its (finite) set of atoms;
- homomorphisms between finite BAs are in one-to-one correspondence with functions between their sets of atoms.
- In categorical language: the category $\mathbf{BA}_f$ of finite Boolean algebras with homomorphisms is dually equivalent to the category $\mathbf{Set}_f$ of finite sets with functions.

The direction of the arrows is reversed!

- However, infinite Boolean algebras cannot — in general — be reconstructed from their atoms.
- Let $\mathbf{F}(\omega)$ be the free Boolean algebra on countably infinite generators. For any $\varphi \in \mathbf{F}(\omega)$ different from $\perp$, pick a variable p which does not appear in φ. Then

  $$\perp < \varphi \wedge p < \varphi.$$

  Thus, $\mathbf{F}(\omega)$ has no atoms!
- We need a generalisation of the concept of atom. This is provided by the notion of ultrafilter:

A subset U of a Boolean algebra B is called an **ultrafilter** provided:

1. $1 \in U$;
2. if $a \in U$ and $b \in B$ satisfies $a \leq b$, then $b \in U$;
3. $a, b \in U$ implies $a \wedge b \in U$;
4. for every $a \in B$, either $a \in U$ or $\neg a \in U$ (but not both).

- Ultrafilters are maximal consistent theories or, equivalently, evaluations.
- They can be thought of as "ideal elements". In the case of finite BAs, they can always be realised as concrete elements:

Lemma (exercise!)
Let B be a finite Boolean algebra. A subset $U \subseteq B$ is an ultrafilter iff

$$U = {\uparrow}x = \{b \in B \mid x \leq b\}$$

for some $x \in \mathbf{At}(B)$.

- As soon as B is infinite, there is always (!) an ultrafilter which is not **principal**, i.e. not of the form ${\uparrow}x$ for any $x \in B$.
  This requires some form of the axiom of choice. Exercise: use Zorn's Lemma to show that there exists an ultrafilter U on $\wp(\mathbb{N})$ which does not contain any finite subset of $\mathbb{N}$. Conclude that U is not principal.

The crucial insight of mathematician M. H. Stone (1934) was that, for every Boolean algebra B, the collection

$$X_B = \{U \subseteq B \mid U \text{ is an ultrafilter}\}$$

carries a **natural topology**. This is generated by the sets of the form

$$\widehat{a} = \{U \in X_B \mid a \in U\}, \text{ for } a \in B.$$

- Logical compactness implies the topological compactness of X_B.
- Due to the presence of negation in the logic, every set $\widehat{a}$ is **clopen** (i.e. both $\widehat{a}$ and its complement are open).
- X_B is a **Boolean (Stone) space**, that is a compact Hausdorff space with a basis of clopen subsets.

- The map

$$B \to \wp(X_B),\ a \mapsto \widehat{a} = \{U \in X_B \mid a \in U\}$$

  is an injective Boolean algebra homomorphism. Its image is the Boolean algebra of clopen subsets of X_B.
- In other words, like in the finite case, **a formula can be identified with the (clopen) set of all worlds in which it is satisfied**.

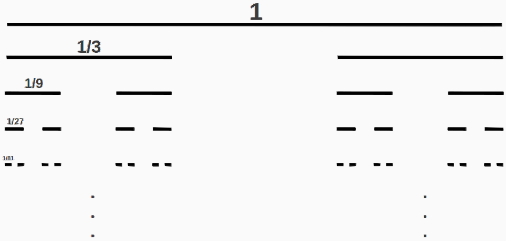
This leads to **full Stone duality for Boolean algebras**:

- every BA is isomorphic to the Boolean algebra of clopen subsets of its space of ultrafilters;
- BA homomorphisms are in one-to-one correspondence with continuous functions between their spaces of ultrafilters (in the opposite direction!).
- In categorical language: the category **BA** of Boolean algebras with homomorphisms is dually equivalent to the category **BStone** of Boolean spaces with continuous maps.

Algebras	dual to	Spaces
subalgebras	$\leftrightarrow$	quotient spaces
homomorphic images	$\leftrightarrow$	closed subspaces
homomorphisms	$\leftrightarrow$	continuous functions
directed unions	$\leftrightarrow$	projective limits
atoms	$\leftrightarrow$	isolated points
countable	$\leftrightarrow$	second-countable

Two examples

- What is the dual space of $\mathbf{F}(\omega)$?
- Evaluations with domain $\mathbf{F}(\omega)$ are in one-to-one correspondence with points of $\{0,1\}^{\omega}$. The Stone topology in this case coincides with the product topology.
- Another description: **Cantor's middle thirds construction**.



$$A_1 = [0,1] \setminus (\tfrac{1}{3}, \tfrac{2}{3}) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$
$$A_2 = A_1 \setminus ((\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})) = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$

- The **Cantor set** is defined as

$$\mathcal{C} = \bigcap_{i \in \mathbb{N}} A_i, \text{ with the subspace topology.}$$

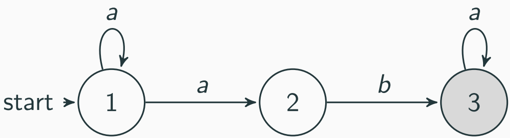
- $\mathcal{C} \neq \emptyset$ because $[0,1]$ is compact. Further,

$$\psi \colon \{0,1\}^\omega \to \mathcal{C},\ (a_i)_{i \in \mathbb{N}} \mapsto \sum_{i \in \mathbb{N}} \frac{2a_i}{3^i}$$

  is a homeomorphism.
- Hence, the Cantor set is dual to $\mathbf{F}(\omega)$.
- Tarski proved that, up to isomorphism, $\mathbf{F}(\omega)$ is the unique non-trivial countable Boolean algebra with no atoms.
- Back in 1910 (when Tarski was 9) Brouwer had shown that, up to homeomorphism, $\mathcal{C}$ is the unique non-trivial Boolean space which is second-countable and has no isolated points.

These two statements are dual to each other!

- The second example is from theoretical computer science.
- **Formal language theory** (Chomsky 1950s) is concerned with the specification and manipulation of sets of strings of symbols.
- For a finite set A (the alphabet), an A-word is an element of the free monoid A^*, and an A-language is a subset of A^*.
- A language is **regular** if it can be recognised by a finite automaton.



- The automaton above recognises the language a^*aba^*.

- An important operation on languages is the concatenation:

$$\forall K, L \in \wp(A^*), \quad KL = \{uv \in A^* \mid u \in K,\ v \in L\}.$$

- This gives rise to an adjoint family of binary operations on $\wp(A^*)$, called residuals:

$$\forall J, K, L \in \wp(A^*), \quad KJ \subseteq L \ \Leftrightarrow\ J \subseteq K\backslash L \ \Leftrightarrow\ K \subseteq L/J.$$

- E.g., $K\backslash L = \{w \in A^* \mid \forall u \in K,\ uw \in L\}$.
- If $K = \{w\}$, the language $\{w\}\backslash L$ is denoted by $w^{-1}L$, the left quotient of L with respect to w. Similarly for right quotients.

- Given a regular language L, the Boolean algebra

$$\mathcal{B}(L) = <\{x^{-1}Ly^{-1} \in \wp(A^*) \mid x, y \in A^*\}>_{BA}$$

is closed under the binary operations $\backslash, /$.
- $\mathcal{B}(L)$ is finite: taking the quotient $w^{-1}L$ amounts to moving the initial state (of a finite automaton recognising L) along the paths labeled by w.
- The dual space of $\mathcal{B}(L)$ is the quotient of A^* by the equivalence relation

$$u \sim_L v \;\Leftrightarrow\; \forall x, y \in A^*(xuy \in L \Leftrightarrow xvy \in L).$$

- Under extended Stone duality, the binary operations $\backslash, /$ on $\mathcal{B}(L)$ correspond to the monoid multiplication inherited by A^*.That is, $\mathcal{B}(L)$ is dual to the so-called **syntactic monoid** of the language L.
- One immediate consequence of the duality is the following well-known fact: a language is regular iff its syntactic monoid is finite.
- Can we **use duality to study arbitrary languages**? See the 2nd part of the tutorial!

References for Part I:

- **General audience**: Gehrke, *Duality*. Oratie (inaugural lecture), Radboud Universiteit, Nijmegen, 2009.
- Book reference for **order/lattice theory and a glimpse of duality**: Davey and Priestley, *Introduction to Lattices and Order*, 2nd edition, Cambridge University Press, 2002.
- Book reference for **duality**: Gehrke and van Gool, *Duality*, forthcoming.
- **Stone's original paper**: Stone, *Boolean Algebras and Their Application to Topology*, PNAS, 1934.
- For the duality theoretic perspective on **formal language theory**, see the introduction to: Gehrke, Petrisan and L. R., *Quantifiers on languages and codensity monads*, LICS 2017.

Part II

Quantifiers and definability

Selected topics in duality

- Part I: Between syntax and semantics
  - Logical validity and truth
  - Stone duality for Boolean algebras
  - Some examples
- Part II: Quantifiers and definability
  - Logic on words
  - Uniform interpolation for IPC

- G.-C. Rota: Indiscrete thoughts (1996)

a. Every lecture should make only one main point

The German philosopher G. W. F. Hegel wrote that any philosopher who uses the word "and" too often cannot be a good philosopher. I think he was right, at least insofar as lecturing goes. Every lecture should state one main point and repeat it over and over, like a theme with variations. An audience is like a herd of cows, moving slowly in the direction they are being driven towards. If we make one point, we have a good chance that the audience will take the right direction; if we make several points, then the cows will scatter all over the field. The audience will lose interest and everyone will go back to the thoughts they interrupted in order to come to our lecture.

My main point: **syntax and semantics are dual to each other**.
Algebras of formulas correspond to spaces of models.

- Duality provides a mathematical language to study the link between syntax and semantics.
- This is useful in mathematical logic (classical, intuitionistic, modal logics...), but also in logic in theoretical computer science.
- It often provides a different perspective on problems. Stone's motto "One must always topologize".
- In some cases it allows us to translate problems from algebra to topology, and back. (E.g., Brouwer's and Tarski's theorems).

Logic on words

- Duality has proved to be a powerful tool in the study of propositional logics. This motivates the study of these methods in the setting also of **logics with quantifiers** such as first-order logic.
- A general framework, based on categories as semantic models for quantified logics, has been developed (Lawvere, Makkai and Reyes,...), but applications are few and far in between.
- We take a bottom-up approach to duality methods in logics with quantifiers, and study quantification in language theory via **logic on words**.

A word $w = w_0 \cdots w_k \in A^*$ is thought of as a structure over $\{0, \ldots, k\} \subseteq \mathbb{N}$ equipped minimally with a unary predicate Q_a (for each $a \in A$) which holds in position $j \in \{0, \ldots, k\}$ iff $w_j = a$.

- This is a study of finite coloured linear orders.
- Every (first-order, or higher-order) sentence φ (interpretable over words) defines a language $L_\varphi = \{w \in A^* \mid w \vDash \varphi\}$.
- $\exists x(\forall z(z \geq x) \wedge Q_a x)$ defines the language aA^*.
- A formula $\varphi(x)$ can be interpreted in **marked words** of the form $(w_0, 0)(w_1, 0)(w_2, 1) \in (A \times 2)^*$.

Theorem (Büchi, 1966)

A language is regular iff it is definable by a monadic second-order sentence using the successor relation.

- Consider a Boolean algebra $\mathcal{B} \subseteq \wp((A \times 2)^*)$ closed under quotients, i.e. $\forall L \in \mathcal{B}$ and $v, w \in (A \times 2)^*$, $v^{-1}Lw^{-1} \in \mathcal{B}$.
- We are interested in the Boolean algebra closed under quotients $\mathcal{B}_\exists \subseteq \wp(A^*)$ generated by $\{L_\exists \mid L \in \mathcal{B}\}$. Here

$$L_\exists = \{w \in A^* \mid \exists 0 \leq i \leq k \text{ s.t. } (w_0, 0) \cdots (w_i, 1) \cdots (w_k, 0) \in L\}.$$

If L is defined by $\varphi(x)$, then $L_\exists$ is defined by $\exists x.\varphi(x)$.

- We want to **understand the construction dual to**

$$\mathcal{B} \longmapsto \mathcal{B}_\exists$$

- To make the expression "dual to" precise one has to take into account (a generalisation of) the algebraic notion of **language recognition**.

Recall that, for any regular language L, the dual of the Boolean algebra

$$\mathcal{B}(L) = <\{x^{-1}Ly^{-1} \in \wp(A^*) \mid x, y \in A^*\}>_{BA}$$

is the syntactic monoid of L.

- If $\mathcal{B} \subseteq \wp(A^*)$ is a Boolean algebra closed under quotients, the residuals $\backslash, /$ dualise to a (continuous) monoid operation iff each $L \in \mathcal{B}$ is regular.
- In general, the dual space always admits continuous actions of a dense subset equipped with a monoid structure.
- These actions are induced by those of the monoid A^* on the dual space of $\wp(A^*)$, that is the Stone-Čech compactification $\beta(A^*)$.

- In joint work with Gehrke and Petrisan we showed that the transformation dual to $\mathcal{B} \mapsto \mathcal{B}_\exists$ essentially amounts to taking a topological semidirect product of X (the dual space of $\mathcal{B}$) and its Vietoris hyperspace $\mathcal{V}(X)$.
- This topological semidirect product is akin to the classical Schützenberger product for monoids.
- More generally, instead of $\exists$, we can consider **semiring quantifiers**. They count the number of witnesses for a formula in a given semiring.
- If S is a finite commutative semiring, we dually get a topological semidirect product with a space of **finitely additive S-valued measures**.

To sum up:

- Applying a layer of semiring quantifiers in logic on words corresponds to a topological semidirect product involving a space of measures.

Why is this interesting?

- It extends to arbitrary languages the powerful algebraic tools available in the context of regular languages.
- It provides further evidence for the applicability of Stone duality in more algorithmic areas of computer science. While in semantics this has long been known (cf. Abramsky's *Domain theory in logical form*, 1991), the deep connection between language theory and duality has started to emerge only around 2008 (Gehrke, Grigorieff, Pin).
- Ongoing joint work with Gehrke and Jakl is showing that, through the lens of duality, quantification in logic on word is related to the study of structural limits, a very active area of **finite model theory** (Nešetřil, de Mendez, ...).

Uniform interpolation for IPC

- Uniform interpolation is a strong property that a (propositional) logic may, or may not, have. It states that certain propositional quantifiers are definable.
- The uniform interpolation property for the intuitionistic propositional calculus was proved by Pitts in 1992.
- In joint work with van Gool we proved an Open Mapping Theorem for the spaces dual to finitely presented Heyting algebras.
- In turn, this yields a short semantic proof of Pitts' uniform interpolation theorem, via Esakia duality for Heyting algebras.

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Volume 57, Number 1, March 1992

ON AN INTERPRETATION OF SECOND ORDER QUANTIFICATION IN FIRST ORDER INTUITIONISTIC PROPOSITIONAL LOGIC

ANDREW M. PITTS

Abstract. We prove the following surprising property of Heyting's intuitionistic propositional calculus, IpC. Consider the collection of formulas, ϕ, built up from propositional variables $(p, q, r, \ldots)$ and falsity $(\perp)$ using conjunction $(\wedge)$, disjunction $(\vee)$ and implication $(\rightarrow)$. Write $\vdash \phi$ to indicate that such a formula is intuitionistically valid. We show that for each variable p and formula ϕ there exists a formula $A_p\phi$ (effectively computable from ϕ), containing only variables not equal to p which occur in ϕ, and such that for all formulas ψ not involving p, $\vdash \psi \rightarrow A_p\phi$ if and only if $\vdash \psi \rightarrow \phi$. Consequently quantification over propositional variables can be modelled in IpC, and there is an interpretation of the second order propositional calculus, IpC^2, in IpC which restricts to the identity on first order propositions.

An immediate corollary is the strengthening of the usual interpolation theorem for IpC to the statement that there are least and greatest interpolant formulas for any given pair of formulas. The result also has a number of interesting consequences for the algebraic counterpart of IpC, the theory of Heyting algebras. In particular we show that a model of IpC^2 can be constructed whose algebra of truth-values is equal to any given Heyting algebra.

If $\varphi(\overline{p}, v)$ is a formula of IPC, a **right uniform interpolant** for φ (wrt v) is a formula $\varphi_R(\overline{p})$ such that, for any formula $\psi(\overline{p}, \overline{q})$ not containing v,

$$\varphi \vdash_{\mathsf{IPC}} \psi \iff \varphi_R \vdash_{\mathsf{IPC}} \psi.$$

Similarly, a **left uniform interpolant** for φ (wrt v) is a formula $\varphi_L(\overline{p})$ such that, for any formula $\psi(\overline{p}, \overline{q})$ not containing v,

$$\psi \vdash_{\mathsf{IPC}} \varphi \iff \psi \vdash_{\mathsf{IPC}} \varphi_L.$$

- Intuition: $\varphi_R = (\exists v)\varphi$ and $\varphi_L = (\forall v)\varphi$

Theorem (Pitts, 1992)
Every formula of IPC admits both right and left uniform interpolants.

- The motivation behind Pitts' result:

The results presented in this paper have had a rather long gestation period. Some ten or so years ago I tried to prove the negation of Theorem 1 in connection with the higher order analogue of Proposition 18—the question of whether any Heyting algebra can appear as the algebra of truth-values of an elementary topos. I established that the free Heyting algebra on a countable infinity of generators does not so appear provided the property of IpC given in Theorem 1 does *not* hold. It seemed likely to me (and to others to whom I posed the question) that a first order proposition ϕ could be found for which $A_p\phi$ does not exist (although I could not find one!), thus settling the original question about toposes and Heyting algebras in the negative. That Theorem 1 is true is quite a surprise to me. Unfortunately, it appears that not all the results for second order logic reported here generalize to the setting of higher order logic. Whilst it is the case that Theorem 1 remains true if IpC is replaced by a quantifier-free fragment of intuitionistic higher order logic, the substitution property of Lemma 8 fails (so that one does not get an interpretation of full higher order logic in its quantifier-free fragment). It remains an open question whether every Heyting algebra can be the Lindenbaum algebra of a theory in intuitionistic higher order logic.

The algebraic understanding of Pitts' result

If $\varphi(\overline{p}, v)$ is a formula of IPC, a right uniform interpolant for φ (wrt v) is a formula $\varphi_R(\overline{p})$ such that, for any formula $\psi(\overline{p}, \overline{q})$ not containing v,

$$\boxed{\varphi \vdash_{\mathsf{IPC}} \psi \iff \varphi_R \vdash_{\mathsf{IPC}} \psi.}$$

By Craig's interpolation theorem for IPC, we can assume that ψ names only variables in $\overline{p}$. Then the monotone map

$$\mathbf{F}_{\mathsf{HA}}(\overline{p}, v) \to \mathbf{F}_{\mathsf{HA}}(\overline{p}), \quad \varphi \mapsto \varphi_R$$

is lower adjoint to the inclusion $\mathbf{F}_{\mathsf{HA}}(\overline{p}) \hookrightarrow \mathbf{F}_{\mathsf{HA}}(\overline{p}, v)$. Similarly, left uniform interpolants correspond to an upper adjoint for $\mathbf{F}_{\mathsf{HA}}(\overline{p}) \hookrightarrow \mathbf{F}_{\mathsf{HA}}(\overline{p}, v)$.

- The uniform interpolation property of IPC is equivalent to the existence of both upper and lower adjoints to the natural inclusion maps ('adding one propositional variable') between free finitely generated Heyting algebras.
- In the case of the classical propositional calculus (more generally, of a locally finite variety with amalgamation), right and left uniform interpolants for $\varphi(\overline{p}, v)$ are given, respectively, by

$$\varphi_R = \bigwedge \{\psi(\overline{p}) \mid \varphi \vdash_{\mathsf{CPC}} \psi\}$$

and

$$\varphi_L = \bigvee \{\psi(\overline{p}) \mid \psi \vdash_{\mathsf{CPC}} \varphi\}.$$

- Ghilardi and Zawadowski (1995, 2002) observed that from Pitts' result it follows that the first-order theory of Heyting algebras admits a model completion. They proved (in fact, for a larger class of logics) that the existence of such a model completion is equivalent to the fact that $(\mathbf{HA}_{fp})^{\mathrm{op}}$ is a Heyting category.
  Further, they showed that this Heyting category structure is induced by restricting the natural Heyting structure of a topos:

$$(\mathbf{HA}_{fp})^{\mathrm{op}} \hookrightarrow \mathbf{Sh}(\mathbf{C})$$

- An algebraic translation of the right uniform interpolation property (in the presence of deductive interpolation) was recently provided by Kowalski and Metcalfe:

Theorem (Kowalski & Metcalfe, Ann. Pure Appl. Logic, to appear)
A variety $\mathcal{V}$ admits right uniform interpolation iff $\mathcal{V}$ is **coherent** *and admits deductive interpolation.*

Historically inaccurate recap on Esakia duality

Our topological approach to Pitts' theorem requires **Esakia duality** for Heyting algebras:

- Esakia duality for Heyting algebras can be seen as a refinement of Priestley duality for (bounded) distributive lattices.
- Stone (1938): distributive lattices and lattice homomorphisms are dually equivalent to spectral spaces and spectral maps.
- The latter duality didn't gain popularity for two reasons: spectral spaces are not Hausdorff and spectral maps are not just continuous maps.
- Priestley (1970): things are way smoother if we work within the class of Nachbin's ordered compact spaces.

An ordered compact space is a pair $(X, \leq)$ where X is a compact space and $\leq \subseteq X \times X$ is a partial order which is closed in the product topology.

- Every ordered compact space is Hausdorff, since the diagonal $\Delta = \leq \cap \geq$ is closed.

A **Priestley space** is an ordered compact space which is totally order-disconnected: whenever $x \not\leq y$, there is a clopen up-set in X that contains x but not y.

For any distributive lattice L, the set X_L of prime filters on L (partially ordered by inclusion) is a Priestley space when equipped with the topology generated by the sets

$$\widehat{a} = \{x \in X_L \mid a \in x\} \text{ and } \widehat{a}^c = \{x \in X_L \mid a \notin x\}, \text{ for } a \in L.$$

- The map

$$L \to \wp(X_L),\ a \mapsto \widehat{a} = \{x \in X_L \mid a \in x\}$$

  is an injective lattice homomorphism. Its image is the distributive lattice of clopen up-sets of X_L.

This leads to **Priestley duality for distributive lattices**:

- every distributive lattice is isomorphic to the lattice of clopen up-sets of its space of prime filters;
- lattice homomorphisms are in one-to-one correspondence with continuous monotone functions between their spaces of prime filters (in the opposite direction!).
- In categorical language: the category **DL** of (bounded) distributive lattices with homomorphisms is dually equivalent to the category **Pries** of Priestley spaces with continuous monotone maps.

Next step: restrict Priestley duality to Heyting algebras.

- Useful fact: Heyting algebras are precisely those distributive lattices L for which the embeddings into their Booleanizations $L \hookrightarrow L^-$ admit upper adjoints.

Lemma (exercise in duality theory)
Let $h\colon L \to M$ be a morphism in **DL**, *and $h^*\colon X_M \to X_L$ the dual morphism in* **Pries**. *The following hold:*

1. *h has a lower adjoint iff $\uparrow h^*(S)$ is open whenever $S \subseteq X_M$ is a clopen up-set;*
2. *h has an upper adjoint iff $\downarrow h^*(S)$ is open whenever $S \subseteq X_M$ is a clopen down-set.*

Taking the Booleanization of L dually amounts to forgetting the order. Thus, L is a Heyting algebra iff $\downarrow S$ is clopen whenever $S \subseteq X_L$ is clopen.

- An **Esakia space** is a Priestley space in which $\downarrow S$ is clopen whenever $S \subseteq X_L$ is clopen.
- The lattice homomorphisms between Heyting algebras which preserve the implication $\to$ correspond precisely to those continuous maps $f : X \to Y$ of Esakia spaces satisfying

$$\forall S \subseteq Y,\ \uparrow f^{-1}(S) = f^{-1}(\uparrow S).$$

These are called p-morphisms, and they are automatically monotone.

Theorem (Esakia, 1974)
Priestley duality restricts to a duality between Heyting algebras with homomorphisms and Esakia spaces with continuous p-morphisms.

- The functors describing the duality are the same: in one direction we take prime filters and in the other one clopen up-sets.

Theorem (van Gool, L. R.)

Every continuous p-morphism between finitely copresented Esakia spaces is an open map.

Proof of Pitts' uniform interpolation theorem.

The right uniform interpolation property holds iff, for every finite set of propositional variables $\overline{p}, v$, the inclusion homomorphism

$$h\colon \mathbf{F}_{\mathsf{HA}}(\overline{p}) \hookrightarrow \mathbf{F}_{\mathsf{HA}}(\overline{p}, v)$$

has a lower adjoint. In turn, by Esakia duality, this is equivalent to saying that the dual morphism

$$h^*\colon E(\overline{p}, v) \twoheadrightarrow E(\overline{p})$$

satisfies S clopen up-set $\Rightarrow {\uparrow}h^*(S) = h^*(S)$ open.

This follows at once from the Open Mapping Theorem. A similar argument applies to settle the left uniform interpolation property. $\square$

- The two previous topics, namely quantification in logic on words and uniform interpolation for IPC, sit somewhere in between propositional and predicate logic. Yet, they admit a nice duality theoretic treatment.
- Another example is Rasiowa and Sikorski's proof of Gödel's completeness theorem for first-order logic using Baire category theorem (1952).
- Other interesting generalisations of propositional logics are obtained by adding infinitary connectives. For instance, there is an extension $\mathcal{L}$ of Łukasiewicz logic (in a language obtained by adding a connective of countably infinite arity) such that the Beth definability property of $\mathcal{L}$ is equivalent to the classical Stone-Weierstrass theorem for compact Hausdorff spaces. (In the sense that the two statements are dual to each other).

References for Part II:

- Duality and **quantification in logic on words**: Gehrke, Petrisan and L. R., *The Schützenberger product for syntactic spaces*, ICALP 2016 and *Quantifiers on languages and codensity monads*, LICS 2017.
- **Pitts' original paper on uniform interpolation for IPC**: Pitts, *On an interpretation of second-order quantification in first-order intuitionistic propositional logic*, JSL 1992.
- A **topological proof of uniform interpolation** for IPC: van Gool and L. R., *An open mapping theorem for finitely copresented Esakia spaces*, Top. Appl., 2018.
- Topological proof of **Gödel's completeness theorem**: Rasiowa and Sikorski, *A proof of the completeness theorem of Gödel*, JSL 1952.
- **Infinitary logic and definability**: L. R., *Beth definability and the Stone-Weierstrass theorem*, forthcoming.