On Injectivity and Projectivity of MV-semimodules

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Around 1958 C. C. Chang devised MV-algebras to study Łukasiewicz logic, in particular MV-algebras form the algebraic semantics of Łukasiewicz logic.

Definition (MV-algebra)
An MV-algebra is an algebra \((A, \oplus, *, 0)\) of type \((2,1,0)\) such that, for every \(x, y, z \in A\) we have:

1. \((x \oplus y) \oplus z = x \oplus (y \oplus z)\);
2. \(x \oplus y = y \oplus x\);
3. \(x \oplus 0 = x\);
4. \((x^*)^* = x\);
5. \(x \oplus 0^* = 0^*\);
6. \((x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x\).
Examples of MV-algebras

Example

1. The real interval $[0, 1]$ with the operations $\oplus$ and $\ast$ defined by $x \oplus y = \min(1, x + y)$ and $x^\ast = 1 - x$ is an MV-algebra. This algebra is called "the standard MV-algebra" and it is very special because it generates the variety of MV-algebras.

2. For every $n \in \mathbb{N}$

$$\mathbb{L}_n := \{0, 1/(n - 1), \ldots, (n - 2)/(n - 1), 1\}$$

yields an MV-algebra with the operations defined as the restriction of the standard MV-algebra of these operations.
On every MV-algebra $A$, it is possible to define another constant $1 := 0^*$ and the operation $x \odot y := (x^* \oplus y^*)^*$. From now on we shall include these in the signature of the MV-algebra, denoting $A$ with $(A, \oplus, \odot, ^*, 0, 1)$. 
For any MV-algebra $A$, there exists a natural order given by:

$$x \leq y \iff x^* \oplus y = 1,$$

for every $x, y \in A$.

The natural order determines a structure of bounded distributive lattice on $A$, with 0 and 1 respectively bottom and top element and the operations of $sup$ and $inf$ defined by:

$$x \lor y := (x \odot y^*) \oplus y$$

and

$$x \land y := (x^* \lor y^*)^*.$$
The Boolean Centre of an MV-algebra

Definition
Let $A$ be an MV-algebra. The boolean centre of $A$ $B(A)$ is the set of all the elements $x \in A$ such that $x \oplus x = x$.

Remark
$B(A)$ is a subalgebra of $A$. 
Some useful results about MV-algebras

Proposition
Let $A$ be an MV-algebra. $A$ is complete and its boolean center $B(A)$ is atomic (as a lattice) iff $A$ is a direct product of complete MV-chains.

Proposition
Every complete MV-chain is either isomorphic to $L_n$ for some $n \in \mathbb{N}$ or to the standard MV-algebra $[0, 1]$. 
Additively idempotent and commutative semirings

Definition (Additively idempotent commutative semiring)

An *additively idempotent commutative semiring* $S$ is an algebra $(S, +, \cdot, 0, 1)$ of type $(2,2,0,0)$ such that:

1. $(S, +, 0)$ is a commutative monoid;
2. $(S, \cdot, 1)$ is a monoid;
3. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$, for every $x, y, z \in S$.
4. $0 \cdot s = 0 = s \cdot 0$, for each $s \in S$;
5. $s + s = s$, for every $s \in S$;
6. $x \cdot y = y \cdot x$, for every $x, y \in S$. 
Definition (MV-semiring)
Let \( S = (S, +, \cdot, 0, 1) \) be a commutative, additively idempotent semiring. \( S \) is a **MV-semiring** iff exists a map \( * : S \to S \), called **negation**, such that:

1. \( a \cdot b = 0 \text{ iff } b \leq a^* \);
2. \( a + b = (a^* \cdot (a^* \cdot b)^*)^* \).

From now on we shall include the negation symbol in the signature of the MV-semiring, denoting \( S \) with \( (S, +, \cdot, 0, 1, *) \).
Theorem

Let $A$ be an MV-algebra. Then $A^{\vee\odot} = (A, \vee, \odot, 0, 1)$ and $A^{\wedge\oplus} = (A, \wedge, \oplus, 1, 0)$ are semirings and the involution $*: A \to A$ is an isomorphism between them. In particular $A^{\vee\odot}$ and $A^{\wedge\oplus}$ are MV-semirings with negation $*$. 

Theorem

If $(A, +, \cdot, 0, 1, *)$ is an MV-semiring, the structure $(A, \oplus, \cdot, *, 0, 1)$ with, for all $x, y \in A$

$$x \oplus y = (x^* \cdot y^*)^*$$

is an MV-algebra.
The previous propositions define a couple of functors which make the categories of MV-algebras and MV-semirings isomorphic.

This allows us to import results and techniques from semiring and ring theory in the study of MV-algebras.

The theory of modules (resp., semimodules) is an essential chapter in the theory of rings (resp., semirings).

We focused on the study of injective and projective MV-semimodules.
Semimodules

Definition (Semimodule)

Let $S$ be a semiring. A (left) $S$ -- semimodule is a commutative monoid $(M, +, 0)$ with a scalar multiplication

$\cdot : (a, x) \in S \times M \rightarrow a \cdot x \in M$, such that the following conditions hold for all $a, b \in S$ and $x, y \in M$:

1. $(ab) \cdot x = a \cdot (b \cdot x)$;
2. $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
3. $(a + b) \cdot x = (a \cdot x) + (b \cdot x)$;
4. $0_S \cdot x = 0_M = a \cdot 0_M$;
5. $1 \cdot x = x$. 
Theorem
Let $S$ be an additively idempotent semiring and $M$ a left $S$-semimodule. Then, $M$ is injective if and only if there exists a set $X$ such that $M$ is a retract of the left $S$-semimodule $\text{Hom}_B(S, B)^X$, where $B$ is the Boolean semifield.

Proposition
Let $S$ be an additively idempotent commutative semiring and $M$ an $S$-semimodule. Then, $M$ is injective iff there exists a set $X$ such that $M$ is a retract of the $S$-semimodule $\text{Id}(S)^X$ (the ideals of $S$ seen as a join-semilattice).
Injective semimodules - Preliminary results

Since we are interested in commutative semirings, we can keep in mind the second characterization of injective semimodules that seems (at least to me) easier to deal with and we can obtain the following

Corollary

Let $S$ be an additively idempotent (and commutative) semiring and $M$ an injective left $S$-semimodule. Then $M$ is a complete semimodule. If in addition the join-semilattice $S$ is a distributive lattice, then $M$ is a complete and infinitely distributive semimodule.
As regards projective semimodules, we have that projective objects are the retracts of the free ones. Given a semiring $S$ and a set $X$ we know that the free $S$-semimodule generated by $X$ is the set of functions from $X$ to $S$ with finite support that we denote with $S^{(X)}$. So, we have the following

**Proposition**

Let $S$ be a semiring and $M$ a $S$-semimodule. Then $M$ is projective iff it is a retract of $S^{(X)}$ for some set $X$. 

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Lemma

(1) For each integer $n \geq 2$, $L_n \cong id(L_n)$ as $L_n$-semimodules. Consequently, $L_n$ is a self-injective semiring.

(2) $[0, 1]$ is a self-injective semiring.

Lemma

Let $S = \prod_{i \in I} S_i$ be a direct product of semirings $S_i$. Then $S$ is left self-injective if and only if each $S_i$ is left self-injective.
Theorem

For any MV-algebra $A$ with an atomic Boolean center, the following conditions are equivalent:

1. The semiring $A^{\lor \otimes}$ is self-injective;
2. All finitely generated projective $A^{\lor \otimes}$-semimodules are injective;
3. $A$ is a complete MV-algebra.
References


 References

1. A. Di Nola, G. Lenzi, T. G. Nam and S. Vannucci, On injectivity of semimodules over additively idempotent division semirings and chain MV-semirings, accepted for publication.


THANK YOU FOR YOUR ATTENTION!