PhDs in Logic 2019

Clause Set Cycles and Induction

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Joint work with Stefan Hetzl

Background: Induction

Induction in Computer Science and Mathematics

Computer Science:

- Automation of proof by induction
- Structural induction (inductive datatypes)
- Algorithms, efficiency, implementation

Mathematics:

- Arithmetical theories
- Consistency, completeness, ...
- Negative results

$\rightsquigarrow$ Independent development

Background: Inductive theorem proving

Methods

- Integration into superposition provers
  - Kersani and Peltier 2013, Σ_1 Induction
  - Cruanes 2017, Σ_1 Induction (?)
  - Wand and Weidenbach 2017, Σ_1 Induction (?)
- Cyclic proofs
- Term rewriting induction
- Theory exploration, Π_1 Induction (?),
- ...

Dimensions

- Analyticity
- Quantifier complexity
- Induction schemes / Orders.

Motivation: Research Program

Automated inductive theorem proving

- Numerous approaches
- Heuristics
- Empirical analysis
- Few formal results

Central questions:

1. What can a given method prove?
2. Why does a method work well?
3. How do methods relate to each other?

$\rightsquigarrow$ Proof theoretic formalization and analysis of methods

Goals: This Talk

Clause Set Cycles inspired by n-clause calculus (NCC) [1]

- analysis tool
- abstracts details
- refutational

Goals

- Describe the induction captured by the system

  Quantifier complexity of induction invariants
- Bound Tightness

  Is the upper bound optimal?
- Completeness

[1] Kersani and Peltier 2013

Outline

Clause Set Cycles

Many-sorted classical first-order logic

- sort nat / symbols $0 : \mathrm{nat}, \mathrm{s} : \mathrm{nat} \to \mathrm{nat}$ / parameter $\mathrm{n} : \mathrm{nat}$

Definition (Clause Set Cycle)

A clause set cycle (CSC) is a clause set $C(\mathrm{n})$ satisfying

$$C(\mathrm{sn}) \models C(\mathrm{n}), \tag{1}$$
$$C(0) \models . \tag{2}$$

Definition

A clause set $S(\mathrm{n})$ is refutable in CSC if there exist a CSC $C(\mathrm{n})$ and $i \in \mathbb{N}$ s.t.

$$S(0) \models, \ \ldots, \ S(\overline{i-1}) \models, \text{ and } S(\mathrm{s}^i\mathrm{n}) \models C(\mathrm{n}). \tag{3}$$

Outline

Σ_1-Upper Bound

Theorem (Σ_1-Bound, CSC $\subseteq \Sigma_1$I)

If a clause set is refutable in CSC, *then it is refutable in the first-order theory of* Σ_1*-induction.*

Σ_1-Upper Bound

Theorem (Σ_1-Bound, CSC $\subseteq \Sigma_1$I)

If a clause set is refutable in CSC, then it is refutable in the first-order theory of Σ_1-induction.

Proof Sketch.

$$
\dfrac{
\begin{array}{c}\vdots\\ S(0) \Rightarrow\end{array}
\quad \ldots \quad
\begin{array}{c}\vdots\\ S(\overline{i-1}) \Rightarrow\end{array}
\qquad
\dfrac{
\begin{array}{c}\vdots\\ S(\mathsf{s}^i\mathsf{n}) \Rightarrow C(\mathsf{n})\end{array}
\qquad
\dfrac{
\begin{array}{c}\vdots\\ \Rightarrow (C(0))^{\neg}\end{array}
\quad
\begin{array}{c}\vdots\\ (C(\mathsf{n}))^{\neg} \Rightarrow (C(\mathsf{sn}))^{\neg}\end{array}
}{\Rightarrow (C(\mathsf{n}))^{\neg}}\ \text{ind}
}{S(\mathsf{s}^i\mathsf{n}) \Rightarrow}\ \text{cut}
}{S(\mathsf{n}) \Rightarrow}\ \text{case}
$$

$\square$

Σ_1-Upper Bound

Theorem (Σ_1-Bound, CSC $\subseteq \Sigma_1$I)
If a clause set is refutable in CSC, then it is refutable in the first-order theory of Σ_1-induction.

Proof Sketch.

$$\dfrac{\begin{matrix}\vdots\\ S(0)\Rightarrow\end{matrix} \quad \dots \quad \begin{matrix}\vdots\\ S(\overline{i-1})\Rightarrow\end{matrix} \quad \dfrac{\begin{matrix}\vdots\\ S(\mathrm{s}^i\mathrm{n})\Rightarrow C(\mathrm{n})\end{matrix} \quad \dfrac{\begin{matrix}\vdots\\ \Rightarrow (C(0))^\neg\end{matrix} \quad \begin{matrix}\vdots\\ (C(\mathrm{n}))^\neg \Rightarrow (C(\mathrm{sn}))^\neg\end{matrix}}{\Rightarrow (C(\mathrm{n}))^\neg}\,\text{ind}}{S(\mathrm{s}^i\mathrm{n})\Rightarrow}\,\text{cut}}{S(\mathrm{n})\Rightarrow}\,\text{case}$$

□

$\rightsquigarrow$ Restriction to Σ_1-induction due to clause normal form.

Outline

Upper Bound Tightness

Theorem ($\mathrm{CSC} \not\subseteq \mathrm{QFI}$)

There exists a clause set which is refutable in CSC, *but is not refutable in the first-order theory of quantifier-free induction.*

Upper Bound Tightness

Theorem (CSC $\not\subseteq$ QFI)

There exists a clause set which is refutable in CSC, *but is not refutable in the first-order theory of quantifier-free induction.*

Let $C = D \vee \neg \mathsf{T}(\mathsf{n}, y)$ with D consisting of

$$x{+}0 = x$$
$$x{+}\mathsf{s}(y) = \mathsf{s}(x{+}y)$$
$$\mathsf{T}(0, 0)$$
$$\mathsf{T}(x, y) \rightarrow \mathsf{T}(\mathsf{s}(x), \mathsf{s}x{+}y).$$

$C(\mathsf{n}) \approx$ "There is no n-th triangular number"

Upper Bound Tightness

Proposition

The clause set C is not refutable in QFI*.*

Proof.

- Find finite axiomatization T' of open induction (similar to Shoenfield 1958)
- Assume $T', C \models \forall n \exists m T(n, m)$
- Get witnesses $t_1(\mathrm{n}), \ldots, t_k(\mathrm{n})$
- Witnesses describe linear functions
- $T(\cdot, \cdot)$ is quadratic. $\square$

Outline

Incompleteness

Conjecture (Σ_1-incompleteness, QFI $\not\subseteq$ CSC $\subset$ $\Sigma 1$I)

There exists a clause set that is refutable in the first-order theory of Σ_1-induction and is not refutable in CSC.

Incompleteness

Conjecture (Σ_1-incompleteness, QFI $\not\subseteq$ CSC $\subset$ Σ1I)

There exists a clause set that is refutable in the first-order theory of Σ_1-induction and is not refutable in CSC*.*

Let $S(\mathsf{n})$ consist of the clauses

$$x + 0 = x \qquad (+^0)$$

$$x + \mathsf{s}(y) = \mathsf{s}(x + y) \qquad (+^{\mathsf{s}})$$

$$\mathsf{n} + (\mathsf{n} + \mathsf{n}) \neq (\mathsf{n} + \mathsf{n}) + \mathsf{n}.$$

Incompleteness

Conjecture (Σ_1-incompleteness, QFI $\not\subseteq$ CSC $\subset$ Σ1I)

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$$\mathrm{n}+(\mathrm{n}+\mathrm{n}) \neq (\mathrm{n}+\mathrm{n})+\mathrm{n}.$$

Lemma

The set $S(\mathrm{n})$ is refutable in QFI.

Proof Sketch.

Prove $x + (x + y) = (x + x) + y$ by induction. $\square$

Incompleteness

Conjecture

The clause set $S(\mathsf{n})$ is not refutable in CSC.

Show that for all $i \in \mathbb{N}$ there is no $\exists$DNF $R(\mathsf{n})$ s.t.

$$\models R(0) \quad \text{(base)}$$

$$R(\mathsf{n}) \models R(\mathsf{sn}) \quad \text{(step)}$$

$$+^{0}, +^{\mathsf{s}}, R(\mathsf{n}) \models \mathsf{s}^i\mathsf{n} + (\mathsf{s}^i\mathsf{n} + \mathsf{s}^i\mathsf{n}) = (\mathsf{s}^i\mathsf{n} + \mathsf{s}^i\mathsf{n}) + \mathsf{s}^i\mathsf{n}. \quad \text{(cut)}$$

Approach: Bottom Up

Incompleteness

($\checkmark$) Disjunctions of literals:
- reduces to single atom case

($\checkmark$) Conjunction of literals:
- restrict structure of atoms
- depth of subterms $\mathsf{s}^k\mathsf{n} + t$ unbounded

($\sim$) Existentially quantified conjunction of literals
- existential witness property
- induces recurrence relation over $\mathbb{N}$
- restrict shape of atoms and witness terms

($\sim$) $\exists$DNF
- negations disappear
- disjunction property
- reduces to $\exists$ conjunction of literals.

Conclusion

- Automated inductive theorem proving
  Benefits from proof theoretical analysis
- Analysis of approaches

  Better understanding, negative results
- Clause Set Cycles

  Abstracts n-clause calculus
- Upper bound, bound tightness, and incompleteness

  Describe the provable sentences