A Sahlqvist theorem for subordination algebras

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Subordination algebras have been studied under several different names: precontact algebras ([7]), proximity algebras ([8]), strict implications algebras ([3]) or even quasi-modal algebras ([4]). The equivalences between all these definitions are well discussed in [2] and [5].

Modal algebras and subordination algebras share a common characteristic: their duals, in the sense of the Stone Duality, are Boolean topological spaces with a closed relation and, in particular, Kripke frames. Therefore the latter denomination, quasi-modal, is not a surprise. Moreover, since subordination algebras form a more general class than modal algebra, we propose an investigation of modal logic through subordination algebras.

While this problem has already been studied under other perspectives, for instance in [1] or in [3], our approach is slightly different as we are interested in validity of modal formulas in subordinations algebras instead of validity of subordination formulas.

Definition 1 (see [2]). A subordination algebra (or quasimodal algebra) is a pair $B = (B, \prec)$ where $B$ is a Boolean algebra and $\prec$ is a binary relation on $B$, called subordination, verifying the following properties:

1. $0 \prec 0$ and $1 \prec 1$,
2. $a \prec b, c$ implies $a \prec b \land c$,
3. $a, b \prec c$ implies $a \lor b \prec c$,
4. $a \leq b \prec c \leq d$ implies $a \prec d$.

Let us remark that a modal algebra $(B, \Diamond)$ may be considered as a subordination algebra $(B, \prec_\Diamond)$ by defining

$$a \prec_\Diamond b \iff \Diamond a \leq b.$$ 

On the other hand, if $B$ is a subordination algebra such for every subset

$$\prec(a, -) := \{b \in B \mid a \prec b\}$$

there is an element $\Diamond a \in B$ such that $\prec(a, -) = \uparrow \Diamond a$, then $(B, \Diamond)$ is a modal algebra.

Definition 2. A subordination space is a pair $X = (X, R)$ where $X$ is a Stone space and $R$ is a binary closed relation on $X$.

The next theorem will not only help us to construct the canonical extension of a subordination algebra but also allow us to prove that subordination spaces play a role for subordination algebras similar to the one played by descriptive frames for modal algebras.

Theorem 3 (see [4]). The category $\text{Sub}$, whose objects are subordination algebras and whose morphisms are the $q$-homomorphisms defined in [4], and the category $\text{SubS}$, whose objects are subordination spaces and whose morphisms are the $q$-morphisms defined in [4], are dually equivalent.
To be seen as models for modal logic, we need to define valuation and validity on subordination algebras. The problem is that we cannot extend freely the valuation for variables to modal formulas, as for instance $\Box p$ may fail to be a clopen set of the dual. In order to resolve this issue, we will focus on the canonical extension of a subordination algebra.

**Theorem 4.** If $B = (B, \prec)$ is a subordination algebra, then its canonical extension $B^d = (\mathcal{P}(X_B), \prec_R)$, where $(X_B, R)$ is the subordination space dual to $B$ and $\prec_R$ is defined by

$$E \prec_R F \iff R(\neg, E) \subseteq F,$$

is a complete modal algebra with $\Diamond E = R(\neg, E)$.

Let us note that the definition of $\prec_R$ in (1) may be considered as arbitrary as we could have also used

$$E \prec_R F \iff R(E, \neg) \subseteq F.$$

This alternative definition will actually lead us to another modal operator denoted by $\boxdot$. With this in mind, we have that $B^d$ is a tense bimodal algebra (see [9]).

**Definition 5.** Let $B$ be a subordination algebra. A valuation on $B$ is a map $v : \text{Var} \rightarrow B$, where Var is the set of variables. In particular, this map can be considered as a map $v : \text{Var} \rightarrow B^d$ and, as such, extend to a bimodal morphism between the set of all bimodal formulas and $B^d$. As usual, we will say that a formula $\varphi$ is valid in $B$ under the valuation $v$, which will be denoted by $B \models v \varphi$, if $v(\varphi) = 1$, where 1 is the top element of both $B$ and $B^d$. The formula $\varphi$ is valid in $B$ if $B \models v \varphi$ for all valuations $v$, this is denoted by $B \models \varphi$.

We will now define a class of bimodal formulas which correspond to first order formulas, in the sense that for a bimodal formula $\varphi$ there is a first order formula $\mathfrak{f}$ such that, for every subordination algebra $B$, we have $B \models \varphi$ if and only if $X_B \models \mathfrak{f}$. The definition of this class of formulas is quite similar to the definition of Sahlqvist formulas for modal algebras (see for instance [10]).

**Definition 6.** Let $\varphi$ be a bimodal formula.

1. It is closed (resp. open) if it is obtained from propositional variables, negation of propositional variables, $\top$ and $\bot$ by applying $\land$, $\lor$, $\Diamond$ and $\Box$ (resp. $\lozenge$ and $\square$).
2. It is positive (resp. negative) if it is obtained from propositional variables (resp. negation of propositional variables) $\top$ and $\bot$ by applying $\land$, $\lor$, $\Diamond$, $\Box$, $\lozenge$ and $\square$.
3. It is strongly positive if it is a conjunction of formulas of the form

$$\Box^{(k)} p = \Box^{k_1} \Box^{k_2} \ldots \Box^{k_n} p$$

where $p$ is a propositional variable, $n \in \mathbb{N}$ and $k \in \mathbb{N}^n$.
4. It is $s$-positive (resp. $s$-negative) if it is obtained from closed positive formulas (resp. open negative formulas) by applying $\land$, $\lor$, $\Box$ and $\square$ (resp. $\Diamond$ and $\lozenge$).
5. It is $s$-untied if it is obtained from strongly positive and $s$-negative formulas by applying $\land$, $\lor$, $\Diamond$ and $\lozenge$.

**Theorem 7.** Let $\varphi = \Box^{(k)} (\varphi_1 \rightarrow \varphi_2)$ be a bimodal formula where $\varphi_1$ is $s$-untied and $\varphi_2$ $s$-positive. Then, $\varphi$ correspond to a first order formula.
References