On Injectivity and Projectivity of MV-semimodules

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MV-semirings are a special class of idempotent semirings strictly connected to MV-algebras, the algebraic semantics of Lukasiewicz propositional logic. In particular, in [3] Di Nola and Russo show that the two aforementioned categories are isomorphic. This fact allows us to import results and techniques from semiring and ring theory into the study of MV-algebras. It is well-known that an effective way to study rings is to study the way in which a ring R acts on its modules. Thus, the theory of modules may be expected to be an essential chapter in the theory of rings. Two of the most important objects in the theory of modules are projective and injective modules ([1]) and, for this reason, we focused on a possible characterization of injective and projective MV-semimodules (semimodules on MV-semirings). In the abstract we report some results of [2].

An MV-algebra is an algebra \((A, \oplus, *, 0)\) of type \((2,1,0)\) such that \((A, \oplus, 0)\) is a commutative monoid; \((x*)* = x; x \oplus 0* = 0*\) and \((x * \oplus y)* \oplus y = (y * \oplus x)* \oplus x\) for every \(x, y \in A\). A semiring is an algebra \((S, +, \cdot, 0, 1)\) of type \((2,2,0,0)\) such that \((S, +, 0)\) is a commutative monoid, \((S, \cdot, 1)\) is a monoid and for all \(x, y, z \in S\) \(x \cdot (y + z) = (x \cdot y) + (x \cdot z), (x + y) \cdot z = (x \cdot z) + (y \cdot z)\) and \(0 \cdot x = x \cdot 0 = 0\). If the semiring \(S\) is idempotent (i. e. \(x + x = x\) for all \(x \in S\)), then \((S, +, 0)\) is a join-semilattice. A MV-semiring is a commutative, idempotent semiring \((S, +, \cdot, 0, 1)\) for which exists a map \(* : S \to S\), called negation, such that: \(a \cdot b = 0\) if \(b \leq a^*\) and \(a + b = (a^* \cdot (a^* \cdot b)^*)^*\).

**Proposition 1.** ([3]) MV-semirings and MV-algebras are isomorphic categories.

Let \((S, +, \cdot, 0, 1)\) be a semiring. A \((left) S\)-semimodule is a commutative monoid \((M, +, 0)\) with a scalar multiplication \(\cdot : A \times M \to M\), denoted \(a \cdot x\), such that the following conditions hold for all \(a, b \in S\) and \(x, y \in M\):

1. \((a \cdot b) \cdot x = a \cdot (b \cdot x)\);
2. \(a \cdot (x + y) = (a \cdot x) + (a \cdot y)\);
3. \((a + b) \cdot x = (a \cdot x) + (b \cdot x)\);
4. \(0_A \cdot x = 0_M = a \cdot 0_M\);
5. \(1 \cdot x = x\).

Right semimodules are defined similarly. If the semiring is commutative left and right semimodules coincide and, since we are only interested in commutative semirings, we will refer generically to semimodules. Any semiring \(S\) is obviously a semimodule over itself. We say that \(S\) is self-injective if it is injective as a semimodule over itself.
Theorem 1. For any MV-algebra $A$ with an atomic Boolean center, the following conditions are equivalent:

1. The semiring $A^\vee \odot$ is self-injective;
2. All finitely generated projective $A^\vee \odot$-semimodules are injective;
3. All cyclic projective $A^\vee \odot$-semimodules are injective;
4. $A$ is a complete MV-algebra.

Theorem 2. Let $A$ be a finite MV-algebra and $M$ a finitely generated $A^\vee \odot$-semimodule. Then the following statements are equivalent:

1. $M$ is injective;
2. $M$ is a retract of a $A^\vee \odot$-semimodule $(A^\vee \odot)^X_X$ for some finite set $X$;
3. $M$ is projective.

Proposition 2. For every MV-algebra $A$, the following statements are equivalent:

1. Every principal ideal of $A^\vee \odot$ is injective;
2. $A^\vee \odot$ is a self-injective and von Neumann regular semiring;
3. $A$ is a complete Boolean algebra.

References