On reducibility notions on the Scott domain

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Abstract

We study reducibility notions on the Scott domain – the set $\mathcal{P}(\mathbb{N})$ endowed with the topology generated by the basis $\big\{\{x \subseteq \mathbb{N} \mid F \subseteq x\} \mid F \subseteq \mathbb{N} \text{ finite}\big\}$. On the one hand, we show that the quasi-order $\leq_{\mathcal{F}_1}$ induced by continuous function on the $\mathbf{\Delta}^0_2$-subsets of the Scott domain is ill-founded and has infinite antichains. On the other hand, we show that the quasi-order $\leq_{\mathcal{F}_2}$ induced by $\mathbf{\Delta}^0_2$-measurable functions on the Borel subsets of the Scott domain is well-founded and has antichains of maximal size 2.

If $\mathcal{X}$ is a topological space, then $\mathcal{F} \subseteq \{f : \mathcal{X} \to \mathcal{X}\}$ is a *reducibility notion* if it contains the identity and is closed under composition. In this case, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, $\mathcal{A}$ is *$\mathcal{F}$-reducible* to $\mathcal{B}$, written $\mathcal{A} \leq_{\mathcal{F}} \mathcal{B}$, whenever there exists a function $f \in \mathcal{F}$ such that $f^{-1}(\mathcal{B}) = \mathcal{A}$. If $\mathcal{F}$ is a reducibility notion, the binary relation $\leq_{\mathcal{F}}$ is a quasi-order that measures the topological complexity of the subsets of a topological space.

We are interested in reducibility notions for the Scott domain:

Definition 1. *The* Scott domain *is the set* $\mathcal{P}(\mathbb{N})$ *endowed with the topology generated by the basis* $\big\{\{x \subseteq \mathbb{N} \mid F \subseteq x\} \mid F \subseteq \mathbb{N}, F \textit{ finite}\big\}$.

The Scott domain $\mathcal{P}(\mathbb{N})$ is important for both the descriptive set theory community and the computer science community. Indeed, this space was recently identified as a universal space among the class of quasi-Polish spaces, a class recently introduced by de Brecht [dB13] as a generalization of Polish spaces. The quasi-Polish spaces both admit a reasonable descriptive set theory and contain important examples of non-Hausdorff spaces involved in theoretical computer science.

Thus, it is natural to study reducibility notions for the Scott domain $\mathcal{P}(\mathbb{N})$. We denote by $\mathcal{F}_1$ the set of all continuous functions from $\mathcal{P}(\mathbb{N})$ to itself, i.e.,

$$\mathcal{F}_1 = \Big\{f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) : f^{-1}(\mathcal{U}) \in \mathbf{\Sigma}^0_1 \text{ for all } \mathcal{U} \in \mathbf{\Sigma}^0_1\Big\}.$$

In [Sel05], Selivanov exhibited antichains of size 4 for the quasi-order $\leq_{\mathcal{F}_1}$ on the Borel subsets of the Scott domain. However, the questions whether this quasi-order $\leq_{\mathcal{F}_1}$ is ill-founded and whether it contains infinite antichains were still open. We recently answered positively, showing that these properties already occur at a low level of topological complexity.

Theorem 2 (V.)**.** *If $\mathcal{F}_1$ is the set of all continuous functions, then the Wadge order $\leq_{\mathcal{F}_1}$ is ill-founded and admits infinite antichains among the $\mathbf{\Delta}^0_2$-subsets of the Scott domain.*

In [MRSS15], Motto Ros, Schlicht and Selivanov proved that the induced quasi-order becomes nicer once you consider more general notions of reducibility. More precisely, if we denote by $\mathcal{F}_\omega$ the set of all $\bigcup_{n\in\omega} \mathbf{\Delta}^0_n$-measurable functions from $\mathcal{P}(\mathbb{N})$ to itself, i.e.,

$$\mathcal{F}_\omega = \Big\{ f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) : f^{-1}(\mathcal{U}) \in \bigcup_{n\in\omega} \mathbf{\Delta}^0_n \text{ for all } \mathcal{U} \in \mathbf{\Sigma}^0_1 \Big\},$$

they proved that the quasi-order $\leq_{\mathcal{F}_\omega}$ on the Borel subsets of the Scott domain is well-founded and has no infinite antichain. We show a possible way to improve this bound by considering the class $\mathcal{F}_2$ of all $\mathbf{\Sigma}^0_2$-measurable functions, i.e.,

$$\mathcal{F}_2 = \Big\{ f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) : f^{-1}(\mathcal{U}) \in \mathbf{\Sigma}^0_2 \text{ for all } \mathcal{U} \in \mathbf{\Sigma}^0_1 \Big\}.$$

Formally, $\mathcal{F}_2$ is not a reducibility notion for it is not closed under composition. However, we have:

Theorem 3 (V.)**.** *If $\mathcal{F}$ is a reducibility notion such that $\mathcal{F}_2 \subseteq \mathcal{F}$ – i.e., which contains all $\mathbf{\Sigma}^0_2$-measurable functions – then the quasi-order $\leq_{\mathcal{F}}$ on the Borel subsets of the Scott domain is well-founded and admits maximal antichains of size 2.*

References

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