Duality in Logic

Proof and Computation
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A logic is \((\text{Form}, \vdash)\) with (at the very least)

\[
\vdash \text{ reflexive and transitive}
\]

So

\[
(\text{Form}/(\vdash \cap \neg \vdash), \vdash /(\vdash \cap \neg \vdash)) \text{ is a partially ordered set}
\]

A lattice \((L, \wedge, \vee, 0, 1)\) is a poset with

- a least element 0 and a greatest element 1
- \(x \wedge y = \inf\{x, y\}\) and \(x \vee y = \sup\{x, y\}\)

(but it can also be defined as an equational class of algebras)
Examples

Which are lattices?

![Diagrams of lattice examples]
**Distributive lattices**

A lattice is **distributive** provided

\[
x \land (y \lor z) = (x \land y) \lor (x \land z)
\]
\[
x \lor (y \land z) = (x \lor y) \land (x \lor z)
\]

Lattices from logic are often distributive (substructural logics being a notable exception)

Let \( X \) be a set, then \( \mathcal{P}(X) \), ordered by \( \subseteq \) is a lattice with

\[
\inf\{A, B\} = A \cap B \quad \text{and} \quad \sup\{A, B\} = A \cup B
\]

A typical example of a distributive lattice is a sublattice of \( \mathcal{P}(X) \)
Examples

Which are lattices? Which are distributive?
Lattice ordered algebras

A lattice expansion (LE) is an algebra

\[(A, \land, \lor, 0, 1, \{f_i\}_{i \in I})\]

such that

- \((A, \land, \lor, 0, 1)\) is a lattice
- For each \(i \in I\), \(f_i : A^{n_i} \rightarrow A\) is an \(n_i\)-ary operation

An LE is called a DLE if the underlying lattice is distributive
Examples of lattice ordered algebras

(BA) **Boolean algebra** \((B, \neg)\) (Classical Prop’al Logic)
- Distributive lattice with
- \(\neg x \land x = 0\) and \(\neg x \lor x = 1\)

(HA) **Heyting algebra** \((H, \rightarrow)\) (Intuitionistic Prop’al Logic)
- Distributive lattice with
- \(x \land y \leq z\) iff \(y \leq x \rightarrow z\)

(MA) **Modal algebra** \((B, \neg, \diamond)\) (Modal Logic)
- \((B, \neg)\) Boolean algebra with
- \(\diamond(x \lor y) = \diamond(x) \lor \diamond(y)\) and \(\diamond(0) = 0\)

(RL) **Residuated lattice** \((L, \cdot, \rightarrow, \leftarrow)\)
- Lattice with
- \(x \cdot y \leq z\) iff \(y \leq x \rightarrow z\) iff \(x \leq z \leftarrow y\)
Stone duality connects distributive lattices and certain topological spaces.

In logic this often plays a role in the following directions:

- **Completeness**: Duality helps in obtaining spatial semantics
- **Decidability**: Sometimes the spatial translation of a question in logic is easier to solve.
Duality — The finite (and complete) case

Finite case, $DL_{\text{fin}}$

behaves like $DL^+$

Lattices that satisfy the following properties

- complete
- completely distributive
- having enough completely join-irreducibles

with complete homomorphisms

Discrete duality
Finite lattices and join-irreducibles

\[ x \neq 0 \text{ is join-irreducible iff } x = y \lor z \implies (x = y \text{ or } x = z) \]

Exercise 1: The same is true for meet-irreducibles!
In the Boolean case, the join-irreducibles ($J$) are exactly the atoms ($At$) and the downset lattice ($D$) of the atoms is just the power set ($P$).
**Categorical Duality**

$DL^+$ — (complete and completely) distributive lattices with enough join irreducibles, with (complete) homomorphisms

$POS$ — partially ordered sets with order preserving maps

![Diagram](image)

$\mathcal{D}(J(D)) \cong D$

$J(\mathcal{D}(P)) \cong P$

$h : D \rightarrow E \iff J(D) \leftarrow J(E) : J(h)$

$\mathcal{D}(P) \leftarrow \mathcal{D}(Q) : \mathcal{D}(f) \iff f : P \rightarrow Q$
Duality for maps

\[ h : D \to E \ \text{\(\land\)-pres} \iff \exists h^\flat : E \to D \ \text{lower adjoint} \]

that is, a map \( h^\flat \) so that

\[ e \leq h(d) \iff h^\flat(e) \leq d \]

In this case, we have

\[ h : D \to E \ \text{also \(\lor\)-pres} \iff h^\flat(J(E)) \subseteq J(D) \]

Duality for (complete) homomorphisms between \( DL^+ \)s

\[ h : D \to E \iff h^\flat : J(E) \to J(D) \]
Duality for maps

\[ f : P \rightarrow Q \hspace{1em} \text{order-preserving} \]

\[ \iff \]

\[ f : D(P) \rightarrow D(Q) \quad \vee \quad \text{and } J \text{ pres extention} \]

\[ U \quad \mapsto \quad \downarrow f[U] \]

\[ \implies \]

\[ (f)\# \hspace{0.5em} \text{is a complete homomorphism} \]

Here \((f)\#\) denotes the upper adjoint of \(f\).

Duality for order-preserving maps between posets

\[ f : P \rightarrow Q \quad \iff \quad (f)\# : D(Q) \rightarrow D(P) \]
Sublattices correspond to order-quotients

For a sublattice $D$, the corresponding quasi-order is given by

$$x \preceq y \iff \forall d \in D \ (y \preceq d \implies x \preceq d)$$

$$D \preceq D_1 \times D_2$$

$$J(D) = (J(D_1) \oplus J(D_2))/\preceq$$
BA⁺ subalgebras correspond to partitions

Let \( B = \langle (ab)^*, a(ba)^*, b(ab)^*, (ba)^* \rangle \) be the Boolean subalgebra of \( \mathcal{P}(A^*) \) generated by these four languages.

The corresponding equivalence relation on \( \text{Atoms}(\mathcal{P}(A^*)) = A^* \) gives the partition

\[
\{ (ab)^+, \{\varepsilon\}, (ba)^+, a(ba)^*, b(ab)^*, Z \}
\]

where \( Z \) is the complement of the union of all the others.
Not enough join-irreducibles

\[ D = 0 \oplus (\mathbb{N}^{op} \times \mathbb{N}^{op}) \] has no join irreducible elements whatsoever!
Prime filters are subsets witnessing missing join-irreducibles (i.e. maximal searches for join-irreducibles)

A filter $F \subseteq D$ is a filter provided

- $1 \in F$ and $0 \not\in F$
- $a \in F$ and $a \leq b \in D$ implies $b \in F$
- $a, b \in F$ implies $a \land b \in F$

(Order dual notion is that of an ideal)

A filter $F \subseteq D$ is prime provided, for $a, b \in D$

$$a \lor b \in F \implies (a \in F \text{ or } b \in F)$$

(Order dual notion is that of a prime ideal)
Prime filters

Let $D$ be a DL and $F \subseteq D$. The following conditions are equivalent:

- $F$ is a prime filter
- $D - F$ is a prime ideal
- $\chi_F : D \to 2$ is a lattice homomorphism

If, in addition, $D$ is finite, then the above are equivalent to

- $F = \uparrow p$ where $p \in J(D)$
The Stone representation theorem

Let $D$ be a distributive lattice, $X_D$ the set of prime filters of $D$, then

$$\eta : D \rightarrow \mathcal{P}(X_D)$$

$$a \mapsto \eta(a) = \{ F \in X_D \mid a \in F \}$$

is an injective lattice homomorphism

- It is easy to verify that $\eta$ preserves 0, 1, $\land$, and $\lor$.
- Injectivity uses Stone’s Prime Filter Theorem:

Let $I$ be an ideal of $D$ and $F$ a filter of $D$. If $I \cap F = \emptyset$, then there is a prime filter $F'$ with $F \subseteq F'$ and $I \cap F' = \emptyset$. 
From the representation theorem to duality

Stone’s representation theorem is akin to Cayley’s theorem for groups: great to know that we have captured the notion of a lattice of sets correctly, but then what?

It is only useful for working with $D$ if we know **H OW $D$ sits inside** $\mathcal{P}(X_D)$. In the case of Cayley there is no answer for this, but for Stone there is a very good answer and it relies on Stone’s motto:

\[ One\ must\ always\ topologize! \]
A topological space

is a structure $(X, \tau)$ where $\tau \subseteq \mathcal{P}(X)$ satisfies

- $\emptyset, X \in \tau$
- $U, V \in \tau$ implies $U \cap V \in \tau$
- $C \subseteq \tau$ implies $\bigcup C \in \tau$

(for a logic interpretation, see geometric logic)

Observations:

- $\text{Top}(X) = \{\tau \in \mathcal{P}(\mathcal{P}(X)) : \tau \text{ is a topology}\}$ is a topped intersection structure and thus a complete lattice
- Each $S \in \mathcal{P}(\mathcal{P}(X))$ generates a topology ($S$ is then a subbasis)
- A subbasis $S$ is called a basis if

$$\mathcal{N}_x = \{U \in S \mid x \in U\}$$

is down-directed for each $x \in X$
A topology $\tau$ on $X$ induces a (quasi-)order on $X$ given by

$$x \leq_{\tau} y \iff \forall U \in \tau \ (y \in U \implies x \in U)$$

Observations:

- $\leq_{\tau}$ is always a quasi-order
- All $U \in \tau$ are downsets in the order $\leq_{\tau}$
- $\leq_{\tau}$ is an order iff $\tau$ is a $T_0$ topology
- $\leq_{\tau}$ is trivial iff $\tau$ is a $T_1$ topology
- $\tau$ is Hausdorff iff distinct points can be separated by disjoint opens

$\leq_{\tau}$ as given here is the reverse of the usual order
Compactness

A subset $C \subseteq X$ is **compact** provided for all $S \subseteq \tau$

$$C \subseteq \bigcup S \implies C \subseteq \bigcup S'$$

for some finite $S' \subseteq S$

Some facts:

- The complement of an open set is called **closed**
- If $X$ is compact, then any closed subset of $X$ is also compact
- If $X$ is Hausdorff and $C \subseteq X$ is compact, then $C$ is closed

In non-Hausdorff compact spaces, there may be compacts which are not closed.
Stone duality

Let $D$ be a DL, the **Stone dual** of $D$ is

$$(X_D, \sigma_D)$$

where $\sigma_D = \langle \eta(a) \mid a \in D \rangle$

then $D \cong \text{Im}(\eta) = \text{CompactOpen}(X)$ and

$$h : D \rightarrow E \iff h^{-1} : X_E \rightarrow X_D$$

It is awkward to state what the dual category is ...
Priestley duality

Let $D$ be a DL, the Priestley dual of $D$ is

$$(X_D, \pi_D, \leq_s)$$

where $\pi_D = \langle \eta(a), (\eta(a))^c \mid a \in D \rangle$

Then the image of $\eta$ is characterized as the clopen downsets

The dual category are the spaces that are:
$C = \text{Compact}$
$TOD = \text{totally order disconnected}$:

$$\forall x, y \ (x \not\leq y \implies \exists U \in \text{ClopDown}(X) y \in U \text{ but } x \not\in U)$$

with continuous and order preserving maps
Completeness CPL

For classical propositional logic we have

$$Form/(\vdash \cap \dashv) = \text{Free}_{BA}(Var)$$

and by Stone duality

$$\eta : \text{Free}_{BA}(Var) \rightarrow \mathcal{P}(X)$$

where $X$ is the dual space of $\text{Free}_{BA}(Var)$

For a formula $\varphi$, we have

$$\varphi \not\equiv 1 \iff \eta(\varphi) \neq X$$

$$\iff \exists \ h : \text{Free}_{BA}(Var) \rightarrow 2 \text{ with } h(\varphi) \neq 1$$

$$\iff \exists \ \tilde{v} : Var \rightarrow 2 \text{ with } \tilde{v}(\varphi) \neq 1$$

So CPL is complete with respect to truth table semantics
Completeness IPC

\[\text{Form} / (\vdash \cap \neg) = \text{Free}_{\text{HA}}(\text{Var})\]

The dual space of \(\text{Free}_{\text{HA}}(\text{Var})\) is \((X, \leq, \pi)\) with the property

\[U \subseteq X \text{ clopen} \implies \uparrow U \text{ is again open}\]

This precisely means that

\[\eta : \text{Free}_{\text{HA}}(\text{Var}) \rightarrow \mathcal{D}(X, \leq)\]

is a Heyting algebra homomorphism

Again this unravels to mean IPC is complete with respect to intuitionistic Kripke semantics
Further examples

1. Completeness of modal logic with respect to Kripke semantics is obtained in a similar way; the binary relation is the dual of the modality.

2. Kripke structures, bounded morphisms, and generated subframes the discrete duals of MA⁺s, their morphisms and quotients.

3. Henkin’s proof of completeness for FOL relies on building a model from an ultrafilter then adding witnesses for existential formulas. This second step can be eliminated by applying Baire category theorem to the dual space of the Lindenbaum algebra.

4. Dana Scott build the first model of the lambda-calculus 40 years after its invention, using an inverse limit of finite spaces (which then is a Stone space). Abramsky later gave a general way of solving domain equations based on Priestley duality.