Spectrum problems for structures arising from lattices and rings

Hochster's Theorem for commutative unital rings

Stone duality for bounded distributive lattices

$\ell$-spectra of Abelian $\ell$-groups

The real spectrum of a commutative, unital ring

Spectral scrummage

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The spectrum of a commutative, unital ring

- A proper ideal \( P \) in a commutative, unital ring \( A \) is **prime** if \( A/P \) is a **domain**. Equivalently, \( xy \in P \Rightarrow (x \in P \text{ or } y \in P) \), for all \( x, y \in A \).
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- Endow the set $\text{Spec } A := \{ P \mid P$ is a prime ideal of $A \}$ with the topology whose closed sets are those of the form $\text{Spec}(A, X) = \{ P \in \text{Spec } A \mid X \subseteq P \}$. This is the so-called hull-kernel topology on $\text{Spec } A$. The topological space thus obtained is the (Zariski) spectrum of $A$.

Is there an intrinsic characterization of the topological spaces of the form $\text{Spec } A$?
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A nonempty closed set $F$ in a topological space $X$ is \textbf{irreducible} if $F = A \cup B$ implies that either $F = A$ or $F = B$, for all closed sets $A$ and $B$. 
Spectral spaces

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  However, usually $U, V \in \mathcal{K}(X) \nRightarrow U \cap V \in \mathcal{K}(X)$. 

A spectral space is sober and $\mathcal{K}(X)$ is a basis of the topology of $X$, closed under finite intersection. Taking the empty intersection then yields that $X$ is compact (usually not Hausdorff). $\operatorname{Spec} A$ is a spectral space, for every commutative unital ring $A$ (well known and easy).
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The converse of the above observation holds:

**Theorem (Hochster 1969)**

Every spectral space $X$ is homeomorphic to $\text{Spec } A$ for some commutative unital ring $A$. 
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- On the ring side, just consider **unital ring homomorphisms**.
- On the spectral space side, consider **surjective spectral maps**.
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- In order for that observation to make sense, the morphisms need to be specified.
- On the ring side, just consider unital ring homomorphisms.
- On the spectral space side, consider surjective spectral maps. For spectral spaces $X$ and $Y$, a map $f : X \to Y$ is spectral if $f^{-1}[V] \in \mathcal{K}(X)$ whenever $V \in \mathcal{K}(Y)$. 
The spectrum of a bounded distributive lattice

- A subset $I$ in a bounded distributive lattice $D$ is an ideal of $D$ if $0 \in I$, ($\{x, y\} \subseteq I \Rightarrow x \vee y \in I$), and ($\{x, y\} \cap I \neq \emptyset \Rightarrow x \wedge y \in I$). An ideal $I$ is prime if $I \neq D$ and ($x \wedge y \in I \Rightarrow \{x, y\} \cap I \neq \emptyset$).
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- A subset $I$ in a bounded distributive lattice $D$ is an ideal of $D$ if $0 \in I$, $\{x, y\} \subseteq I \Rightarrow x \lor y \in I$, and $\{x, y\} \cap I \neq \emptyset \Rightarrow x \land y \in I$. An ideal $I$ is prime if $I \neq D$ and $x \land y \in I \Rightarrow \{x, y\} \cap I \neq \emptyset$.

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- It is well known that the spectrum of any bounded distributive lattice is a spectral space.
The functors underlying Stone duality

For bounded distributive lattices $D$ and $E$ and a 0, 1-lattice homomorphism $f : D \to E$, the map $\text{Spec } f : \text{Spec } E \to \text{Spec } D$, $Q \mapsto f^{-1}[Q]$ is spectral.
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- For bounded distributive lattices $D$ and $E$ and a $0,1$-lattice homomorphism $f: D \to E$, the map $\text{Spec } f: \text{Spec } E \to \text{Spec } D$, $Q \mapsto f^{-1}[Q]$ is spectral.

- For spectral spaces $X$ and $Y$ and a spectral map $\varphi: X \to Y$, the map $\hat{\mathcal{K}}(\varphi): \hat{\mathcal{K}}(Y) \to \hat{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$ is a $0,1$-lattice homomorphism.
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Theorem (Stone 1938)

The pair $(\text{Spec}, \mathcal{K})$ induces a (categorical) duality, between bounded distributive lattices with $0,1$-lattice homomorphisms and spectral spaces with spectral maps.
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The pair $(\text{Spec}, \mathcal{K})$ induces a (categorical) duality, between bounded distributive lattices with 0, 1-lattice homomorphisms and spectral spaces with spectral maps.

Note that in Hochster’s Theorem’s case, we do not obtain a duality (a ring is not determined by its spectrum).
To summarize: spectral spaces are the same as spectra of commutative unital rings, and also spectra of bounded distributive lattices.
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- **To summarize**: spectral spaces are the same as spectra of commutative unital rings, and also spectra of bounded distributive lattices.

- In the case of **bounded distributive lattices**, we obtain a duality. In the case of **commutative unital rings**, we do not.

- Further algebraic structures also afford a concept of spectrum.
An \( \ell \)-group is a group endowed with a lattice ordering \( \leq \), such that \( x \leq y \) implies both \( xz \leq yz \) and \( zx \leq zy \).
**$\ell$-ideals of an Abelian $\ell$-group**

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The $\ell$-spectrum of an Abelian $\ell$-group with unit

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The ℓ-spectrum of an Abelian ℓ-group with unit

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- It turns out that more is true!
Completely normal spectral spaces

- In any topological space $X$, the **specialization preordering** is defined by $x \leq y$ if $y \in \{x\}$.
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- A spectral space $X$ is completely normal if $\leq$ is a root system, that is, $\{x, y\} \subseteq \overline{\{z\}} \Rightarrow (x \in \overline{\{y\}} \text{ or } y \in \overline{\{x\}})$. 

\[ \text{Theorem (Monteiro 1954)} \]

A spectral space $X$ is completely normal if its Stone dual $\langle X \rangle$ is a completely normal lattice, that is, $\forall a, b (a \lor b = a \lor y = x \lor b \text{ and } x \land y = 0)$. 

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Theorem (Monteiro 1954)

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$$(\forall a, b)(\exists x, y)(a \lor b = a \lor y = x \lor b \text{ and } x \land y = 0).$$
The $\ell$-spectrum of any Abelian $\ell$-group with unit is a completely normal spectral space.

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\( \ell \)-spectra of Abelian \( \ell \)-groups again

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Not every completely normal spectral space is an \(\ell\)-spectrum.
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**Theorem (Delzell and Madden 1994)**

Not every completely normal spectral space is an \( \ell \)-spectrum.

Delzell and Madden’s example is not second countable (i.e., no countable basis of the topology): in fact, it has
\[
\operatorname{card} \mathcal{K}(X) = \aleph_1.
\]
Every second countable completely normal spectral space is homeomorphic to $\text{Spec}_\ell G$ for some Abelian $\ell$-group $G$ with unit.
Theorem (W. 2017)

Every second countable completely normal spectral space is homeomorphic to \( \text{Spec}_\ell G \) for some Abelian \( \ell \)-group \( G \) with unit.

- Hence, Delzell and Madden’s counterexample cannot be extended to the countable case.
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- Since \(G\) has an order-unit, \(\text{Id}_c G\) is a bounded distributive lattice.
\textbf{\(\ell\)-spectra of countable Abelian \(\ell\)-groups}

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\begin{itemize}
  \item Hence, Delzell and Madden’s counterexample \textbf{cannot} be extended to the \textbf{countable} case.
  \item \textbf{Very rough outline of proof} (of the countable case): start by observing that for any \textbf{Abelian \(\ell\)-group} \(G\) with unit, the \textbf{Stone dual} of \(\text{Spec}_\ell G\) is \(\text{Id}_c G\), the lattice of all \textbf{principal \(\ell\)-ideals} of \(G\) (ordered by \(\subseteq\)).
  \item Since \(G\) has an \textbf{order-unit}, \(\text{Id}_c G\) is a \textbf{bounded distributive lattice}.
  \item Thus we must prove that \textbf{every countable completely normal bounded distributive lattice} \(D\) is \(\cong \text{Id}_c G\) for some \textbf{Abelian \(\ell\)-group} \(G\) with unit.
\end{itemize}
Very rough outline of the proof of the countable case (cont’d)

- The idea is to construct a “nice” surjective 0,1-lattice homomorphism \( f : \text{Id}_c F_\omega \to D \), where \( F_\omega \) denotes the free Abelian \( \ell \)-group on a countably infinite generating set.
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Definition (closed maps)

For bounded distributive lattices $A$ and $B$, a 0,1-lattice homomorphism $f : A \rightarrow B$ is **closed** if whenever $a_0, a_1 \in A$ and $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$. 
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Very rough outline of the proof of the countable case (further cont’d)

- The map \( f : \text{Id}_c F_\omega \to D \) is constructed as \( f = \bigcup_{n < \omega} f_n \) (each \( f_n \subseteq f_{n+1} \)), where each \( f_n : L_n \to D \) is a lattice homomorphism, for a carefully constructed finite sublattice \( L_n \) of \( \text{Id}_c F_\omega \).
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The subset \( \text{Op}^-(\mathcal{H}) = \text{Op}(\mathcal{H}) \setminus \{ \mathbb{E} \} \) is a sublattice of \( \text{Op}(\mathcal{H}) \).
Very rough outline of the proof of the countable case (coming to the end)

- The lattices $L_n$ will have the form $\text{Op}^-(\mathcal{H})$, for finite sets of integer hyperplanes in $\mathbb{E} = \mathbb{R}^{(\omega)}$. 
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Each enlargement step, from $f^n$ to $f^{n+1}$, corrects one of the following three types of defects:

- (hard) $f^n$ is not defined everywhere: then add a pair $(H^+, H^-)$ to the domain of $f^n$;

- (easy, but infinite dimension needed!) $f^n$ is not surjective: then add an element to the range of $f^n$;

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\[ \text{Spectrum problems for structures arising from lattices and rings} \]

\[ \text{Hochster's Theorem for commutative unital rings} \]

\[ \text{Stone duality for bounded distributive lattices} \]

\[ \ell\text{-spectra of Abelian } \ell\text{-groups} \]

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\[ \text{Spectral scrummage} \]
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Analogous result for $\mathcal{L}_{\infty,\lambda}$ (for any infinite cardinal $\lambda$): proof currently under verification.
Cones, prime cones, real spectrum

- The **real spectrum** was introduced in 1981, by Coste and Coste-Roy, as an ordered analogue of the Zariski spectrum of a commutative unital ring.
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- Let $A$ be a commutative unital ring (**not necessarily ordered**). A **cone** of $A$ is a subset $C$ of $A$ such that $C + C \subseteq C$, $C \cdot C \subseteq C$, and $a^2 \in C$ whenever $a \in A$. 
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We endow the set $\text{Spec}_r A$ of all prime cones of $A$ with the topology generated by the sets $\{ P \in \text{Spec}_r A \mid a \notin P \}$, for $a \in A$. 
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- It turns out that $\text{Spec}_r A$ is a completely normal spectral space, for any commutative unital ring $A$. 
Characterizing problem of real spectra

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Theorem (Mellor and Tressl 2012)

For any infinite cardinal $\lambda$, there is no $L_{\infty,\lambda}$-characterization of the Stone duals of real spectra of commutative unital rings.
Subspaces of $\ell$-spectra and real spectra

It is known that every \textbf{closed} subspace of an $\ell$-spectrum (resp., real spectrum) is an $\ell$-spectrum (resp., real spectrum).
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\textbf{Problem (W. 2017)}

Is a \textbf{retract} of an $\ell$-spectrum also an $\ell$-spectrum? Same question for real spectra.
Comparing spectra

- For any class $\mathbf{X}$ of spectral spaces, denote by $\mathbf{SX}$ the class of all spectral subspaces of members of $\mathbf{X}$. 

Then introduce the following classes of spectral spaces:

- $\mathbf{CN}$, the class of all completely normal spectral spaces;
- $\mathbf{ell}$, the class of all $\ell$-spectra of Abelian $\ell$-groups with unit;
- $\mathbf{R}$, the class of all real spectra of commutative unital rings.

**Theorem (W. 2017)**

All containments and non-containments of the following picture are valid:

$$ \mathbf{CN} = \mathbf{SCN} \subseteq \mathbf{S} \subseteq \mathbf{ell} \subsetneq \mathbf{R} $$
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Theorem (W. 2017)

All containments and non-containments of the following picture are valid:

$$CN = SCN$$

$\ell$ $\ell$-spectra of Abelian $\ell$-groups

$CN = SCN$ implies $SR \subset R$.
All the separating counterexamples, intervening in the result above, have size $\aleph_1$, except for the counterexample witnessing $\mathbf{S\ell} \not\subseteq \mathbf{CN}$, which has size $\aleph_2$. 
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Most of the examples constructed for the theorem above involve the construction of condensate (Gillibert and W. 2011), which turns diagram counterexamples to object counterexamples, with a jump of alephs corresponding to the order-dimension of the poset indexing the diagram (thus $\aleph_1$, $\aleph_2$, and so on).
Knebusch and Scheiderer proved in 1989 that for any homomorphism $f: R \to S$ of commutative unital rings, the map $\text{Spec}_R f: \text{Spec}_R S \to \text{Spec}_R R$ is convex, that is, whenever $Q_0 \subseteq Q_1$ in $\text{Spec}_R S$, $P \in \text{Spec}_R R$, and $f^{-1}Q_0 \subseteq P \subseteq f^{-1}Q_1$, there exists $Q \in \text{Spec}_R S$ such that $Q_0 \subseteq Q \subseteq Q_1$ and $P = f^{-1}Q$. Let $K$ be any countable, non-Archimedean real-closed field, and set $A = \{x \in K | (\exists n < \omega)(-n \cdot 1 \leq x \leq n \cdot 1)\}$. The counterexample is the ring $R$ of all almost constant families $(x_\xi | \xi < \omega_1) \in K^{\omega_1}$ such that $x_\infty \in A$: there is no Abelian $\ell$-group $G$ such that $\text{Spec}_R R \cong \text{Spec}_\ell G$. This is partly due to Knebusch and Scheiderer’s result.
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A counterexample witnessing $\text{SR} \not\subseteq R$

- Start with a countable real-closed domain with exactly three prime ideals \( \{0\} \subsetneq P_1 \subsetneq P_2 \). Then consider the ring \( E \) of all almost constant \( \omega_1 \)-indexed families of elements of \( A \).
A counterexample witnessing $\text{SR} \notin \mathbb{R}$

- Start with a countable real-closed domain with exactly three prime ideals $\{0\} \subsetneq P_1 \subsetneq P_2$. Then consider the ring $E$ of all almost constant $\omega_1$-indexed families of elements of $A$.

- Define $\varphi : 4 = \{0, 1, 2, 3\} \overset{\text{def}}{\rightarrow} 3 = \{0, 1, 2\}$ as the Stone dual of the (non-convex) map $\{1, 2\} \rightarrow \{1, 2, 3\}$, $1 \mapsto 1$, $2 \mapsto 3$. Hence $\varphi(0) = 0$, $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = 2$. However, $\text{Cond}(\varphi, \omega_1)$ is a homomorphic image of the dual space of the real spectrum of $E$. 


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- The lattice $\text{Cond}(\varphi, \omega_1) = \{ (x, y) \in 4 \times 3^{\omega_1} \mid y_\xi = \varphi(x) \text{ for all but finitely many } \xi \}$ is not the dual space of any real spectrum (because of Knebusch and Scheiderer’s result).
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A counterexample witnessing $\ell \not\subseteq \text{SR}$

For any chain $\Lambda$, denote by $\mathbb{Z}\langle \Lambda \rangle$ the lexicographical power of $\mathbb{Z}$ by $\Lambda$: hence $\alpha < \beta$ in $\Lambda$ implies that $n\alpha < \beta$ in $\mathbb{Z}\langle \Lambda \rangle$ for every integer $n$. 
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- Denote by $F$ the Abelian $\ell$-group defined by generators $a$ and $b$ subjected to the relations $a \geq 0$ and $b \geq 0$. 
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- The counterexample is the lexicographical product \( G = \mathbb{Z}\langle \omega_1^{\text{op}} \rangle \times_{\text{lex}} F \):

- \( \text{Spec}_\ell G \) cannot be embedded, as a spectral subspace, into the real spectrum of any commutative unital ring.
A counterexample witnessing $\text{CN} \not\subseteq \text{Sl}$

- Start observing that any homomorphic image of the Stone dual of any $\text{Spec}_\ell G$ satisfies the following family of infinitary statements:
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- For any family $(a_i \mid i \in I)$, there are elements $c_{i,j}$ such that each $a_i = (a_i \land a_j) \lor c_{i,j}$, each $c_{i,j} \land c_{j,i} = 0$, and each $c_{i,k} \leq c_{i,j} \lor c_{j,k}$.
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- Consider the variety $\mathcal{V}$, in the similarity type $(0, 1, \lor, \land, \setminus)$, whose identities are those of bounded distributive lattices, together with the additional identities

$$x = (x \land y) \lor (x \setminus y); \quad (x \setminus y) \land (y \setminus x) = 0.$$
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- It works because of Kuratowski’s Free Set Theorem.
Thanks for your attention!