Uniform Interpolation and Proof Systems

Rosalie Iemhoff
Utrecht University

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An old question: When does a logic have a decent proof system?
An old question: When does a logic have a decent proof system?

Another old question: When does a logic have a sequent calculus?
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Answers: Many positive instances. Less negative ones.
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Related work:
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Related work:

(Negri) Fix a labelled sequent calculus and determine which axioms, when added, preserve cut-elimination.
Proof systems

An old question: When does a logic have a decent proof system?

Another old question: When does a logic have a sequent calculus?

Answers: Many positive instances. Less negative ones.

Related work:

(Negri) Fix a labelled sequent calculus and determine which axioms, when added, preserve cut-elimination.

(Ciabattoni, Galatos, Terui) Fix a sequent calculus and determine which axioms or structural rules, when added, preserve cut-elimination.
An old question: When does a logic have a decent proof system?

Another old question: When does a logic have a sequent calculus?

Answers: Many positive instances. Less negative ones.

Related work:

(Negri) Fix a labelled sequent calculus and determine which axioms, when added, preserve cut-elimination.

(Ciabattoni, Galatos, Terui) Fix a sequent calculus and determine which axioms or structural rules, when added, preserve cut-elimination.

Aim: Formulate properties that, when violated by a logic, imply that the logic does not have a sequent calculus of a certain form.
Dfn A logic $L$ has interpolation if whenever $\vdash \varphi \rightarrow \psi$ there is a $\chi$ in the common language $L(\varphi) \cap L(\psi)$ such that $\vdash \varphi \rightarrow \chi$ and $\vdash \chi \rightarrow \psi$. 
Dfn A logic $L$ has interpolation if whenever $\vdash \varphi \rightarrow \psi$ there is a $\chi$ in the common language $L(\varphi) \cap L(\psi)$ such that $\vdash \varphi \rightarrow \chi$ and $\vdash \chi \rightarrow \psi$.

Dfn A propositional (modal) logic has uniform interpolation if the interpolant depends only on the premiss or the conclusion: For all $\varphi$ there are formulas $\exists p \varphi$ and $\forall p \varphi$ not containing $p$ such that for all $\psi$ not containing $p$:

$$\vdash \psi \rightarrow \varphi \iff \vdash \psi \rightarrow \forall p \varphi \quad \vdash \varphi \rightarrow \psi \iff \vdash \exists p \varphi \rightarrow \psi.$$
Dfn A logic $L$ has interpolation if whenever $\vdash \varphi \to \psi$ there is a $\chi$ in the common language $\mathcal{L}(\varphi) \cap \mathcal{L}(\psi)$ such that $\vdash \varphi \to \chi$ and $\vdash \chi \to \psi$.

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$$\vdash \psi \to \varphi \iff \vdash \psi \to \forall p \varphi \quad \vdash \varphi \to \psi \iff \vdash \exists p \varphi \to \psi.$$  

Algebraic view (next talk).
**Uniform interpolation**

**Dfn** A logic $L$ has interpolation if whenever $\vdash \phi \to \psi$ there is a $\chi$ in the common language $L(\phi) \cap L(\psi)$ such that $\vdash \phi \to \chi$ and $\vdash \chi \to \psi$.

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Algebraic view *(next talk)*.

**Note** A locally tabular logic that has interpolation, has uniform interpolation.

\[
\exists p \phi(p, \bar{q}) = \bigwedge \{ \psi(\bar{q}) \mid \vdash \phi(p, \bar{q}) \to \psi(\bar{q}) \}
\]

\[
\forall p \phi(p, \bar{q}) = \bigvee \{ \psi(\bar{q}) \mid \vdash \psi(\bar{q}) \to \phi(p, \bar{q}) \}
\]
**Thm (Pitts '92)** IPC has uniform interpolation.
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**Thm** (Shavrukov '94) GL has uniform interpolation.
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Thm (Maxsimova '77, Ghilardi & Zawadowski '02)
There are exactly seven intermediate logics with (uniform) interpolation:

**IPC, Sm, GSc, LC, KC, Bd₂, CPC.**
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There are exactly seven intermediate logics with (uniform) interpolation:

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Pitts uses Dyckhoff’s '92 sequent calculus for **IPC**.
Aim: If a modal or intermediate logic has such an such a sequent calculus, then it has uniform interpolation.
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Therefore no modal or intermediate logic without uniform interpolation has such an such a calculus.
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*Therefore* no modal or intermediate logic without uniform interpolation has such an such a calculus.

**Modularity:** The possibility to determine whether the addition of a new rule will preserve uniform interpolation.
Dfn Multiplication of sequents:

\[(\Gamma \Rightarrow \Delta) \cdot (\Pi \Rightarrow \Sigma) \equiv (\Gamma, \Pi \Rightarrow \Delta, \Sigma).\]
Focussed rules

**Dfn** Multiplication of sequents:

\[(\Gamma \Rightarrow \Delta) \cdot (\Pi \Rightarrow \Sigma) \equiv (\Gamma, \Pi \Rightarrow \Delta, \Sigma)\].

**Dfn** A rule is focussed if it is of the form

\[
\frac{S \cdot S_1 \ldots S \cdot S_n}{S \cdot S_0}
\]

where \(S, S_i\) are sequents and \(S_0\) contains exactly one formula.
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**Ex** The following rules are focussed.

\[
\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \land B, \Delta}
\]
**Dfn** Multiplication of sequents:

\[(\Gamma \Rightarrow \Delta) \cdot (\Pi \Rightarrow \Sigma) \equiv (\Gamma, \Pi \Rightarrow \Delta, \Sigma).\]

**Dfn** A rule is focussed if it is of the form

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\hline
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\[
\frac{\Gamma, B \rightarrow C \Rightarrow A \rightarrow B \quad \Gamma, C \Rightarrow D}{\Gamma, (A \rightarrow B) \rightarrow C \Rightarrow D}
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**Focussed rules**

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**Ex** The following rules are focussed.

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\frac{\Gamma, B \rightarrow C \Rightarrow A \rightarrow B \quad \Gamma, C \Rightarrow D}{\Gamma, (A \rightarrow B) \rightarrow C \Rightarrow D}
\]

**Dfn** An axiom is focussed if it is of the form

\[
\Gamma, p \Rightarrow p, \Delta \quad \Gamma, \bot \Rightarrow \Delta \quad \Gamma \Rightarrow \top, \Delta \quad \ldots
\]
Dfn A calculus is terminating if there exists a well-founded order on sequents such that in every rule the premisses come before the conclusion, and . . .
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**Thm** Every terminating calculus that consists of focussed axioms and rules has uniform interpolation.
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**Thm** Every terminating calculus that consists of focused axioms and rules has uniform interpolation.

**Cor** Classical propositional logic has uniform interpolation.
**Dfn** A calculus is terminating if there exists a well-founded order on sequents such that in every rule the premisses come before the conclusion, and . . .

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**Cor** Classical propositional logic has uniform interpolation.

**Cor** Intuitionistic propositional logic has uniform interpolation.
Dfn A calculus is terminating if there exists a well-founded order on sequents such that in every rule the premisses come before the conclusion, and . . .

Thm Every terminating calculus that consists of focussed axioms and rules has uniform interpolation.

Cor Classical propositional logic has uniform interpolation.

Cor Intuitionistic propositional logic has uniform interpolation.

Cor Except the seven intermediate logics that have interpolation, no intermediate logic has a terminating sequent calculus that consists of focussed rules and axioms.
Propositional logic

**Dfn** A calculus is terminating if there exists an well-founded order on sequents such that in every rule the premisses come before the conclusion, and . . .
**Propositional logic**

**Dfn** A calculus is *terminating* if there exists an well-founded order on sequents such that in every rule the premisses come before the conclusion, and . . .

**Thm** Every logic with a terminating calculus that consists of focussed axioms and rules has uniform interpolation.
**Propositional logic**

**Dfn** A calculus is terminating if there exists an well-founded order on sequents such that in every rule the premisses come before the conclusion, and . . .

**Thm** Every logic with a terminating calculus that consists of focussed axioms and rules has uniform interpolation.

**Proof idea:**
Define interpolation for rules. For every instance

\[
\frac{S_1 \ldots S_n}{S_0} \quad R
\]

of a rule, define the formula \( \forall^R p S_0 \) in terms of \( \forall p S_i \) \((i > 0)\). For example,

\[
\forall^R p S_0 \equiv \forall p S_1 \land \ldots \land \forall p S_n.
\]
**Propositional logic**

**Dfn** A calculus is terminating if there exists an well-founded order on sequents such that in every rule the premisses come before the conclusion, and...

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Then inductively define

\[
\forall p S \equiv \bigvee \{ \forall^R p S \mid R \text{ is an instance of a rule with conclusion } S \}
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Propositional logic

**Dfn** A calculus is *terminating* if there exists an well-founded order on sequents such that in every rule the premisses come before the conclusion, and ...

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Then inductively define

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For free sequents \( S \) define \( \forall p S \) separately.
**Propositional logic**

**Dfn** A calculus is terminating if there exists an well-founded order on sequents such that in every rule the premisses come before the conclusion, and . . .

**Thm** Every logic with a terminating calculus that consists of focussed axioms and rules has uniform interpolation.

**Proof idea:**
Define interpolation for rules. For every instance

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of a rule, define the formula \(\forall^R p S_0\) in terms of \(\forall p S_i\) (\(i > 0\)). For example,

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Then inductively define

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\forall p S \equiv \bigvee \{\forall^R p S \mid R \text{ is an instance of a rule with conclusion } S\}
\]

*For free sequents* \(S\) *define* \(\forall p S\) *separately.*

Prove with induction on the order that for all sequents \(S\) a uniform interpolant \(\forall p S\) exists.
\[ \text{Dfn } (\Gamma \Rightarrow \Delta)^a = \Gamma \text{ and } (\Gamma \Rightarrow \Delta)^s = \Delta \text{ and } \forall p \phi = \forall p (\Rightarrow \phi). \]
**Propositional logic**

**Dfn** \((\Gamma \Rightarrow \Delta)^a = \Gamma \text{ and } (\Gamma \Rightarrow \Delta)^s = \Delta \text{ and } \forall p \varphi = \forall p (\Rightarrow \varphi)\).

A logic has uniform interpolation if it satisfies the **interpolant properties**:

- (\forall l) for all \(p\): \(\vdash S^a, \forall p S \Rightarrow S^s\);
- (\forall r) for all \(p\): \(\vdash S^l \cdot (\Rightarrow \forall p S^r)\) if \(S^l \cdot S^r\) is derivable and \(S^l\) does not contain \(p\).
**Propositional logic**

*Dfn* \((\Gamma \Rightarrow \Delta)^a = \Gamma\) and \((\Gamma \Rightarrow \Delta)^s = \Delta\) and \(\forall p \varphi = \forall p (\Rightarrow \varphi)\).

*A logic has uniform interpolation if it satisfies the interpolant properties:*

(\(\forall l\)) for all \(p\): \(\vdash S^a, \forall p S \Rightarrow S^s\);  
(\(\forall r\)) for all \(p\): \(\vdash S^l \cdot (\Rightarrow \forall p S^r)\) if \(S^l \cdot S^r\) is derivable and \(S^l\) does not contain \(p\).

*From (\(\forall l\)) obtain \(\vdash \forall p \varphi \Rightarrow \varphi\).*
**Propositional logic**

\[ \text{Dfn} \; (\Gamma \Rightarrow \Delta)^a = \Gamma \; \text{and} \; (\Gamma \Rightarrow \Delta)^s = \Delta \; \text{and} \; \forall p \varphi = \forall p (\Rightarrow \varphi). \]

A logic has uniform interpolation if it satisfies the interpolant properties:

(\forall l) for all \( p \): \( \vdash S^a, \forall p S \Rightarrow S^s \);

(\forall r) for all \( p \): \( \vdash S^l \cdot (\Rightarrow \forall p S^r) \) if \( S^l \cdot S^r \) is derivable and \( S^l \) does not contain \( p \).

From (\forall l) obtain \( \vdash \forall p \varphi \rightarrow \varphi \).

From (\forall r) obtain that \( \vdash \psi \rightarrow \varphi \) implies \( \vdash \psi \rightarrow \forall p \varphi \), if \( \psi \) does not contain \( p \), by taking \( S^l = (\psi \Rightarrow ) \) and \( S^r = (\Rightarrow \varphi) \). Hence \( \vdash \psi \rightarrow \varphi \iff \vdash \psi \rightarrow \forall p \varphi \), if \( \psi \) does not contain \( p \).
Propositional logic

Dfn \((\Gamma \Rightarrow \Delta)^a = \Gamma\) and \((\Gamma \Rightarrow \Delta)^s = \Delta\) and \(\forall p \varphi = \forall p(\Rightarrow \varphi)\).

A logic has uniform interpolation if it satisfies the interpolant properties:

(\forall l) for all \(p\): \(\vdash S^a, \forall p S \Rightarrow S^s\);

(\forall r) for all \(p\): \(\vdash S^l \cdot (\Rightarrow \forall p S^r)\) if \(S^l \cdot S^r\) is derivable and \(S^l\) does not contain \(p\).

From \((\forall l)\) obtain \(\vdash \forall p \varphi \rightarrow \varphi\).

From \((\forall r)\) obtain that \(\vdash \psi \rightarrow \varphi\) implies \(\vdash \psi \rightarrow \forall p \varphi\), if \(\psi\) does not contain \(p\), by taking \(S^l = (\psi \Rightarrow )\) and \(S^r = (\Rightarrow \varphi)\). Hence \(\vdash \psi \rightarrow \varphi \iff \vdash \psi \rightarrow \forall p \varphi\), if \(\psi\) does not contain \(p\).

A logic satisfies the interpolant properties if it satisfies:

(1) \(\{S_j \cdot (\forall p S_j \Rightarrow ) | 1 \leq j \leq n\} \vdash S_0 \cdot (\forall p S_0 \Rightarrow )\).

(2) \(\{S^l_j \cdot (\Rightarrow \forall p S^r_j) | 1 \leq j \leq n\} \vdash S^l_0 \cdot (\Rightarrow \forall p S^r_0)\).

(3) If \(S^r_0\) is no conclusion of \(R\) there exists . . .
Propositional logic

Dfn \((\Gamma \Rightarrow \Delta)^a = \Gamma\) and \((\Gamma \Rightarrow \Delta)^s = \Delta\) and \(\forall p \varphi = \forall p (\Rightarrow \varphi)\).

A logic has uniform interpolation if it satisfies the interpolant properties:

\((\forall l)\) for all \(p\): \(\vdash S^a, \forall p S \Rightarrow S^s\);

\((\forall r)\) for all \(p\): \(\vdash S^l \cdot (\Rightarrow \forall p S^r)\) if \(S^l \cdot S^r\) is derivable and \(S^l\) does not contain \(p\).

From \((\forall l)\) obtain \(\vdash \forall p \varphi \rightarrow \varphi\).

From \((\forall r)\) obtain that \(\vdash \psi \rightarrow \varphi\) implies \(\vdash \psi \rightarrow \forall p \varphi\), if \(\psi\) does not contain \(p\), by taking \(S^l = (\psi \Rightarrow )\) and \(S^r = (\Rightarrow \varphi)\). Hence \(\vdash \psi \rightarrow \varphi \iff \vdash \psi \rightarrow \forall p \varphi\), if \(\psi\) does not contain \(p\).

A logic satisfies the interpolant properties if it satisfies:

\(1\) \(\{S_j \cdot (\forall p S_j \Rightarrow ) \mid 1 \leq j \leq n\} \vdash S_0 \cdot (\forall p S_0 \Rightarrow )\).

\(2\) \(\{S^l_j \cdot (\Rightarrow \forall p S^r_j) \mid 1 \leq j \leq n\} \vdash S^l_0 \cdot (\Rightarrow \forall p S^r_0)\).

\(3\) If \(S^r_0\) is no conclusion of \(R\) there exists . . .

A logic satisfies the above properties if it has a terminating calculus that consists of focussed axioms and rules.
Dfn A focussed modal rule is of the form

\[
\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}
\]

where \(S_1\) and \(S_2\) consist of multisets and \(S_0\) of multisets and exactly one atom.
**Modal logic**

**Dfn** A *focussed modal rule* is of the form

\[
\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}
\]

where \(S_1\) and \(S_2\) consist of multisets and \(S_0\) of multisets and exactly one atom.

**Ex** The following are focussed modal rules.

\[
\begin{align*}
\Gamma \Rightarrow p & \quad \Pi, \Box\Gamma \Rightarrow \Box p, \Sigma & \quad \text{R}_K \\
\Box\Gamma, p \Rightarrow \Box\Delta & \quad \Pi, \Box\Gamma, \Box p \Rightarrow \Box\Delta, \Sigma & \quad \text{R}_{\Box K} \\
p \Rightarrow \Delta & \quad \Pi, \Box p \Rightarrow \Box\Delta, \Sigma & \quad \text{R}_{\Box OK}
\end{align*}
\]
A focussed modal rule is of the form

\[
\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}
\]

where \(S_1\) and \(S_2\) consist of multisets and \(S_0\) of multisets and exactly one atom.

The following are focussed modal rules.

\[
\frac{\Gamma \Rightarrow p}{\Pi, \Box \Gamma \Rightarrow \Box p, \Sigma} \quad \frac{\Box \Gamma, p \Rightarrow \Box \Delta}{\Pi, \Box \Gamma, \Box p \Rightarrow \Box \Delta, \Sigma} \quad \frac{p \Rightarrow \Delta}{\Pi, \Box p \Rightarrow \Box \Delta, \Sigma}
\]

A modal logic with a balanced terminating calculus that consists of focussed axioms and focussed (modal) rules and contains \(R_K\) or \(R_{OK}\), has uniform interpolation.
A focussed modal rule is of the form

\[
\begin{array}{c}
\Box S_1 \cdot S_0 \\
S_2 \cdot \Box S_1 \cdot \Box S_0
\end{array}
\]

where \( S_1 \) and \( S_2 \) consist of multisets and \( S_0 \) of multisets and exactly one atom.

The following are focussed modal rules.

\[
\begin{align*}
\Gamma & \Rightarrow p & R_K \\
\Pi, \Box \Gamma & \Rightarrow \Box p, \Sigma & R_K \\
\Pi, \Box \Gamma & \Rightarrow \Box p, \Sigma & R_K \\
\Pi, \Box \Gamma & \Rightarrow \Box \Delta, \Sigma & R_K \\
p & \Rightarrow \Delta & R_OK \\
p & \Rightarrow \Delta & R_OK \\
p & \Rightarrow \Delta & R_OK \\
\Pi, \Box p & \Rightarrow \Box \Delta, \Sigma & R_K \\
\Pi, \Box p & \Rightarrow \Box \Delta, \Sigma & R_K \\
\Pi, \Box p & \Rightarrow \Box \Delta, \Sigma & R_K
\end{align*}
\]

A modal logic with a balanced terminating calculus that consists of focussed axioms and focussed (modal) rules and contains \( R_K \) or \( R_OK \), has uniform interpolation.

Any normal modal logic with a balanced terminating calculus that consists of focussed (modal) axioms and rules, has uniform interpolation. (Ex: \( K \))
Dfn A **focussed modal rule** is of the form

\[
\begin{array}{c}
\square S_1 \cdot S_0 \\
S_2 \cdot \square S_1 \cdot \square S_0
\end{array}
\]

where \( S_1 \) and \( S_2 \) consist of multisets and \( S_0 \) of multisets and exactly one atom.

Ex The following are focussed modal rules.

\[
\begin{align*}
\Gamma &\Rightarrow p \\
\Pi, \square \Gamma &\Rightarrow \square p, \Sigma \\
\square \Gamma, p &\Rightarrow \square \Delta \\
\Pi, \square \Gamma, \square p &\Rightarrow \square \Delta, \Sigma \\
p &\Rightarrow \Delta \\
\Pi, \square p &\Rightarrow \square \Delta, \Sigma
\end{align*}
\]

R\_K R\_OK

Cor A modal logic with a balanced terminating calculus that consists of focussed axioms and focussed (modal) rules and contains \( R\_K \) or \( R\_OK \), has uniform interpolation.

Cor Any normal modal logic with a balanced terminating calculus that consists of focussed (modal) axioms and rules, has uniform interpolation. (Ex: \( K \))

Cor The logics \( K4 \) and \( S4 \) do not have balanced terminating sequent calculi that consist of focussed (modal) axioms and rules.
Questions

- Extend the method to other modal logics, such as GL and KT.
- Extend the method to hypersequents.
- Use other proof systems than sequent calculi.
Finis