Admissible Rules of (Fragments of) R-Mingle

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R-Mingle

Relevance logic $\mathbf{R}$ with Mingle

$p \rightarrow (p \rightarrow p)$
R-Mingle

Relevance logic R with Mingle

\[ p \to (p \to p) \]

RM with additional constant \( t \)

Language

\[ \mathcal{L}_t = \{ \land, \lor, \to, \cdot, \neg, t \} \]
Definition

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- $\Gamma/\varphi$ is admissible in a logic $L$ if for all substitutions (homomorphisms) $\sigma : Fm_L \rightarrow Fm_L$:
  
  $$\vdash_L \sigma(\psi) \text{ for all } \psi \in \Gamma \Rightarrow \vdash_L \sigma(\varphi)$$
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  \[ \vdash_L \sigma(\psi) \text{ for all } \psi \in \Gamma \implies \vdash_L \sigma(\varphi) \]
- $\{\Gamma/\varphi \mid \Gamma/\varphi \text{ is admissible in } L\} =: \sim_L$
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- Let $R$ be a set of rules. $L + R =$ smallest logic containing $L \cup R$
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- $\{\Gamma/\varphi \mid \Gamma/\varphi \text{ is admissible in } L\} =: \sim_L$
- Let $\mathcal{R}$ be a set of rules.
  $$L + \mathcal{R} = \text{smallest logic containing } L \cup \mathcal{R}$$
- $\mathcal{R}$ is a basis for the admissible rules of $L$ if $L + \mathcal{R} = \sim_L$
Corresponding algebraic semantics

\[ Z^\circ = \langle \mathbb{Z} \setminus \{0\}, \min, \max, \to, \cdot, -, 1 \rangle \]

\[ x \to y := \begin{cases} 
\max\{-x, y\} & \text{if } x \leq y \\
\min\{-x, y\} & \text{if } x > y 
\end{cases} \]

\[ x \cdot y := \begin{cases} 
\min\{x, y\} & \text{if } |x| = |y| \\
y & \text{if } |x| < |y| \\
x & \text{if } |x| > |y| 
\end{cases} \]
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\[ \cdot \quad x \cdot y := \begin{cases} \min\{x, y\} & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |x| > |y| \end{cases} \]

\[ Z_{2n} = \langle \{-n, \ldots, -1, 1, \ldots, n\}, \min, \max, \to, \cdot, -, 1 \rangle \]

\[ Z_{2n+1} = \langle \{-n, \ldots, -1, 0, 1, \ldots, n\}, \min, \max, \to, \cdot, -, 1 \rangle \]
Sugihara Monoids

$SM = \forall (Z^\circ)$ the variety of Sugihara Monoids generated by $Z^\circ$. $SM$ provides an equivalent algebraic semantics for $RM^t$

$$\{\psi \approx |\psi| \mid \psi \in \Gamma\} \models_{SM} \varphi \approx |\varphi| \iff \Gamma \models_{SM} \varphi \iff \Gamma \vdash_{RM^t} \varphi$$

for any rule $\Gamma / \varphi$. 

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Admissible Rules of (Fragments of) R-Mingle
This talk

Bases for admissible rules of the fragments of $\mathsf{RM}^t$ with the following languages

\begin{align*}
\mathcal{L}_1 &= \{\rightarrow, t\} \\
\mathcal{L}_2 &= \{\rightarrow, \cdot, t\} \\
\mathcal{L}_m &= \{\rightarrow, \neg, t\} = \{\rightarrow, \cdot, \neg, t\} \text{ multiplicative fragment.}
\end{align*}
This talk

Bases for admissible rules of the fragments of $\text{RM}^t$ with the following languages

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$\text{SM} \upharpoonright \mathcal{L}_i$ algebraic semantics corresponding to the $\mathcal{L}_i$-fragment of $\text{RM}^t$, $i \in \{1, 2, m\}$
This talk

Bases for admissible rules of the fragments of $\text{RM}^t$ with the following languages

$$L_1 = \{\to, t\}$$

$$L_2 = \{\to, \cdot, t\}$$

$$L_m = \{\to, \neg, t\} = \{\to, \cdot, \neg, t\}$$ multiplicative fragment.

$SM \upharpoonright L_i$ algebraic semantics corresponding to the $L_i$-fragment of $\text{RM}^t$, $i \in \{1, 2, m\}$

Remark $\text{RM}^t \upharpoonright \{\land, \to, t\}$ has empty basis (= it is structurally complete).

Raftery, Olson
Idea of how to find the bases

Recall

\[ SM = \forall(SM) = \forall(Z^\circ) \]
Idea of how to find the bases

Recall

\[ SM = \bigvee(SM) = \bigvee(Z^\circ) \]

Lemma S.

\[ \bigvee(SM \upharpoonright L_i) = \bigvee(Z_4 \upharpoonright L_i), \quad i \in \{1, 2, m\} \]
Idea of how to find the bases

Recall that if for two varieties \( \mathbb{V}_1 \) and \( \mathbb{V}_2 \) we have:
\[
\mathbb{V}_1 = \mathbb{V}_2 \iff (\vdash_{\mathbb{V}_1} \varphi \iff \vdash_{\mathbb{V}_2} \varphi) \text{ for all formulas } \varphi.
\]
Idea of how to find the bases

- Recall that if for two varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ we have:
  $\mathcal{V}_1 = \mathcal{V}_2$ iff $(\models_{\mathcal{V}_1} \varphi \iff \models_{\mathcal{V}_2} \varphi$ for all formulas $\varphi)$.
- A rule is admissible in $\text{RM}^t \upharpoonright \mathcal{L}_i$ if:
  $\iff$ it is admissible in $\text{SM} \upharpoonright \mathcal{L}_i$
  $\iff$ it is admissible in $\text{Z}_4 \upharpoonright \mathcal{L}_i$
Idea of how to find the bases

- Recall that if for two varieties $V_1$ and $V_2$ we have:
  $$V_1 = V_2 \iff (\vdash_{V_1} \varphi \iff \vdash_{V_2} \varphi \text{ for all formulas } \varphi).$$
- A rule is admissible in $RM^t \upharpoonright L_i$ if it is admissible in $SM \upharpoonright L_i$ if it is admissible in $Z_4 \upharpoonright L_i$
- Interested in algebras s.t. admissibility in $Z_4 \upharpoonright L_i$ corresponds to validity in these algebras.
Idea of how to find the bases

- Recall that if for two varieties \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) we have:
  \[ \mathcal{V}_1 = \mathcal{V}_2 \iff (\vdash_{\mathcal{V}_1} \phi \iff \vdash_{\mathcal{V}_2} \phi) \text{ for all formulas } \phi. \]
- A rule is admissible in \( \text{RM}^t \upharpoonright \mathcal{L}_i \) if and only if it is admissible in \( \text{SM} \upharpoonright \mathcal{L}_i \) if and only if it is admissible in \( \mathcal{Z}_4 \upharpoonright \mathcal{L}_i \).
- Interested in algebras s.t. admissibility in \( \mathcal{Z}_4 \upharpoonright \mathcal{L}_i \) corresponds to validity in these algebras.
- Then: Axiomatize the quasivarieties generated by these algebras to get an axiomatization of the admissible rules of our fragments.
Theorem

Let \( \mathcal{B} \) be an algebra and \( \mathcal{F}_\mathcal{B}(\omega) \) its free algebra on countably infinite many generators. Then

\[
\Gamma / \varphi \text{ is } \mathcal{B}\text{-admissible} \iff \Gamma \models_{\mathcal{F}_\mathcal{B}(\omega)} \varphi.
\]
Finding the bases

The following are equivalent:

(i) \( \Gamma/\varphi \) is \( \mathbf{B} \)-admissible \iff \( \Gamma \models_A \varphi \)

(ii) \( Q(A) = Q(F_B(\omega)) \)
Finding the bases

Lemma

The following are equivalent:

(i) $\Gamma/\varphi$ is $B$-admissible $\iff \Gamma \models_{A} \varphi$

(ii) $\mathbb{Q}(A) = \mathbb{Q}(F_{B}(\omega))$

So we want to find $A$ which is “easy” to axiomatize - but how?
Finding the bases

The following are equivalent:

(i) \( \Gamma/\varphi \) is \( \mathcal{B} \)-admissible  \iff  \( \Gamma \models_\mathcal{A} \varphi \)

(ii) \( \mathcal{Q}(\mathcal{A}) = \mathcal{Q}(\mathcal{F}_\mathcal{B}(\omega)) \)

So we want to find \( \mathcal{A} \) which is “easy” to axiomatize - but how?

\( \mathcal{A} \subseteq \mathcal{F}_\mathcal{B}(\omega), \mathcal{B} \in \mathcal{H}(\mathcal{A}) \ \Rightarrow \ \mathcal{Q}(\mathcal{A}) = \mathcal{Q}(\mathcal{F}_\mathcal{B}(\omega)) \)
The algebras in our case

Lemma S.

Let $Z'_4 \subset (Z_2 \times Z_3) \upharpoonright L_1$, $Z''_4 \subset (Z_2 \times Z_3) \upharpoonright L_2$, $(Z_2 \times Z_3) \upharpoonright L_m$ be the algebras pictured. Then

(i) $\mathcal{Q}(F_{Z_4}|L_1(\omega)) = \mathcal{Q}(Z'_4)$

(ii) $\mathcal{Q}(F_{Z_4}|L_2(\omega)) = \mathcal{Q}(Z''_4)$

(iii) $\mathcal{Q}(F_{Z_4}|L_m(\omega)) = \mathcal{Q}((Z_2 \times Z_3) \upharpoonright L_m)$

Figure: $Z'_4$ and $Z''_4$
Definition

\[ |\psi| := \psi \rightarrow \psi \]
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\[ \varphi \Rightarrow \psi := (\varphi \rightarrow |\psi|) \rightarrow (\varphi \rightarrow \psi) \]
Definition

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- $\varphi \Rightarrow \psi := (\varphi \rightarrow |\psi|) \rightarrow (\varphi \rightarrow \psi)$
- $\{p, p \Rightarrow q\}/q$ (A)
Definition

- $|\psi| := \psi \rightarrow \psi$
- $\varphi \Rightarrow \psi := (\varphi \rightarrow |\psi|) \rightarrow (\varphi \rightarrow \psi)$
- $\{p, p \Rightarrow q\}/q$ (A)
- $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \cdot (\psi \rightarrow \varphi)$

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Definition

- $|\psi| := \psi \rightarrow \psi$
- $\varphi \Rightarrow \psi := (\varphi \rightarrow |\psi|) \rightarrow (\varphi \rightarrow \psi)$
- $\{p, p \Rightarrow q\}/q$ \hspace{1cm} (A)
- $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \cdot (\psi \rightarrow \varphi)$
- $\{-(|p_1| \leftrightarrow \ldots \leftrightarrow |p_n|)\}/q$ \hspace{1cm} (R_n), \hspace{0.5cm} n \in \mathbb{N}.$
The Bases

Lemma S.

We have the following axiomatizations:

(i) \[ \text{RM}^t \upharpoonright \mathcal{L}_1 + (A) \text{ has equivalent q.v. } \mathcal{Q}(\mathbb{Z}_4') \]
(ii) \[ \text{RM}^t \upharpoonright \mathcal{L}_2 + (A) \text{ has equivalent q.v. } \mathcal{Q}(\mathbb{Z}_4'') \]
(iii) \[ \text{RM}^t \upharpoonright \mathcal{L}_m + (A) + \{(R_n)\}_{n \in \mathbb{N}} \text{ has eq. q.v. } \mathcal{Q}((\mathbb{Z}_2 \times \mathbb{Z}_3) \upharpoonright \mathcal{L}_m) \]
The Bases

Lemma S. We have the following axiomatizations:

(i) $\text{RM}_t \upharpoonright \mathcal{L}_1 + (A)$ has equivalent q.v. $\mathbb{Q}(\mathbb{Z}_4')$

(ii) $\text{RM}_t \upharpoonright \mathcal{L}_2 + (A)$ has equivalent q.v. $\mathbb{Q}(\mathbb{Z}_4'')$

(iii) $\text{RM}_t \upharpoonright \mathcal{L}_m + (A) + \{(R_n)\}_{n \in \mathbb{N}}$ has eq. q.v. $\mathbb{Q}((\mathbb{Z}_2 \times \mathbb{Z}_3) \upharpoonright \mathcal{L}_m)$

Theorem S. Then as a Corollary of this lemma

(i) $\{(A)\}$ is a basis for the $\{\rightarrow, t\}$- and $\{\rightarrow, \cdot, t\}$-fragment of $\text{RM}_t$.

(ii) $\{(A)\} \cup \{(R_n)\}_{n \in \mathbb{N}}$ is a basis for $\text{RM}_t \upharpoonright \{\rightarrow, \neg, t\}$. 

Our Conjecture

Look again at $RM$ without constant $t$. 
Our Conjecture

Look again at $\text{RM}$ without constant $t$.

$\left(B\right)$

$\{\neg|p| \lor q\}/q$
Our Conjecture

Look again at RM without constant $t$.

$\{\neg \| p \| \lor q \}/q$

We hope to prove the following:

(i) RM + (B) is almost structurally complete, i.e.,

$$\Gamma \models_{RM} \varphi \Rightarrow \Gamma \vdash_{RM} \varphi$$

whenever there is a substitution

$$\sigma : \text{Fm}_{\mathcal{L}} \rightarrow \text{Fm}_{\mathcal{L}}$$

s.t. for all $\psi \in \Gamma$,

$$\vdash_{RM} \sigma(\psi).$$

(ii) The admissible rules of RM have basis

$$\{(B)\} \cup \{(R_n)\}_{n \in \mathbb{N}}.$$
References


